State Estimation for Robotics Tim Barfoot Chapter 3 June 10, 2018

3.6.1 $x_k = x_{k-1} + v_k + w_k \qquad w_k \sim N(0, Q)$

$y_k = x_k + n_k \qquad n_k \sim N(0, R)$

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_K \end{bmatrix} \qquad z = \begin{bmatrix} \cancel{\check{x}_0} \\ v_1 \\ \vdots \\ v_K \\ y_0 \\ \vdots \\ y_K \end{bmatrix} \qquad W = \begin{bmatrix} \cancel{\check{P}_0} & & & & & \\ & Q_1 & & & & \\ & & \ddots & & & \\ & & & Q_K & & \\ & & & & R_0 & \\ & & & & & \ddots \\ & & & & & & R_K \end{bmatrix}$$

note $\check{x}_0$ is unknown

note: $\check{P}_0$ is unknown since $\check{x}_0$ is unknown

$$H = \begin{bmatrix} A^{-1} \\ C \end{bmatrix} = \begin{bmatrix} \cancel{1} & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & & -1 & 1 \\ 1 & & & & \\ & \ddots & & & \\ & & & & 1 \end{bmatrix} \begin{matrix} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} K \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} K+1 \end{matrix}$$

$$H^T = \begin{array}{c c} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} 2 \\ 1 \\ \\ \\ \\ \\ \end{matrix} \end{array}$$

add 1 to 2 and we have
Rank $K+1 = 6$
$\therefore H^T W^{-1} H$ is invertible!
and the problem is solvable

3.6.2 $H^T W^{-1} H$ note $W^{-1} = I$ as $Q = R = 1$

$$= H^T H = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$\boxed{\text{use } H^T \text{ listed on 3.6.1 soln}}$

~~Since $H^T H = L L^T$ we can have $L = H^T$. I am not sure if there is a better soln...~~

See Cholesky decom on next page

Sol'n based on 3.2.2

$L_0 L_0^T = 2 \Rightarrow L_0 = \sqrt{2}$

$L_{10} L_0^T = -1 \Rightarrow L_{10} = -\sqrt{\frac{1}{2}}$

$L_1 L_1^T = -\frac{1}{2} + 3 \Rightarrow L_1 = \sqrt{\frac{5}{2}}$

$L_{21} L_1^T = -1 \Rightarrow L_{21} = -\sqrt{\frac{2}{5}}$

continuing this will give us

$$L = \begin{bmatrix} \sqrt{2} & & & & & \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{5}{2}} & & & & \\ & -\sqrt{\frac{2}{5}} & \sqrt{\frac{13}{5}} & & & \\ & & -\sqrt{\frac{5}{13}} & \sqrt{\frac{34}{13}} & & \\ & & & -\sqrt{\frac{13}{34}} & \sqrt{\frac{89}{34}} & \\ & & & & -\sqrt{\frac{34}{89}} & \sqrt{\frac{144}{89}} \end{bmatrix} \blacksquare$$

3.6.3

$$W = \begin{bmatrix} Q_1 & & & & & & & \\ & \ddots & & & & & & \\ & & Q_K & & & & & \\ & & & R_0 & R/2 & R/4 & 0 & 0 & 0 \\ & & & R/2 & R_1 & R/2 & R/4 & 0 & 0 \\ & & & R/4 & R/2 & R_2 & R/2 & R/4 & \text{[illegible]} \\ & & & 0 & R/4 & R/2 & R_3 & R/2 & R/4 \\ & & & 0 & 0 & R/4 & R/2 & R_4 & R/2 \\ & & & 0 & 0 & 0 & R/4 & R/2 & R_5 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} R & \frac{1}{2}R & \frac{1}{4}R & 0 & 0 & 0 \\ \frac{1}{2}R & R & \frac{1}{2}R & \frac{1}{4}R & 0 & 0 \\ \frac{1}{4}R & \frac{1}{2}R & R & \frac{1}{2}R & \frac{1}{4}R & 0 \\ 0 & \frac{1}{4}R & \frac{1}{2}R & R & \frac{1}{2}R & \frac{1}{4}R \\ 0 & 0 & \frac{1}{4}R & \frac{1}{2}R & R & \frac{1}{2}R \\ 0 & 0 & 0 & \frac{1}{4}R & \frac{1}{2}R & R \end{bmatrix}$$

Since the new $\mathbf{R}$ is still positive def, the same rules for sol'n uniqueness applies, namely if $\mathrm{rank}(H^T) = K+1$ which was shown in 3.6.2 so this problem has a unique sol'n.

3.6.4 KF equations

$$\check{P}_k = \hat{P}_{k-1} + Q \quad ①$$
$$\check{x}_k = \hat{x}_{k-1} + v_k$$
$$K_k = \check{P}_k\left(\check{P}_k + R\right)^{-1}$$

$$\hat{P}_k = \left(1 - \frac{\check{P}_k}{\check{P}_k + R}\right)\check{P}_k = \frac{R}{\check{P}_k + R}\check{P}_k \quad ②$$
$$\hat{x}_k = \check{x}_k + \frac{\check{P}_k}{\check{P}_k + R}\left(y - \check{x}_k\right)$$

Using ① and ② to express the eq. as only $\check{P}_k$ and only $\hat{P}_k$

sub ② in ① : $\check{P}_k = \frac{R}{\hat{P}_{k-1} + R}\hat{P}_{k-1} + Q$

~~As~~ $k \to \infty$, $\check{P}_k = \check{P}_{k-1}$ so let us denote $\check{P}_k$ as $\check{P}$

$$\check{P} = R\left(\check{P} + R\right)^{-1}\check{P} + Q$$
$$\check{P}^2 + \cancel{\check{P}R} = \cancel{R\check{P}} + Q\check{P} + QR$$
$$\check{P}^2 - Q\check{P} - QR = 0$$

sub ① in ② : $\hat{P}_k = \frac{R}{\hat{P}_{k-1} + Q + R}\hat{P}_{k-1} + Q$

Similarly, as $k \to \infty$, $\hat{P}_k = \hat{P}_{k-1}$ so let us denote $\hat{P}_k$ as $\hat{P}$

$$P = R\left(\hat{P} + Q + R\right)^{-1}\left(\hat{P} + Q\right)$$
$$\hat{P}^2 + Q\hat{P} + \cancel{R\hat{P}} = \cancel{R\hat{P}} + RQ$$
$$\hat{P}^2 + Q\hat{P} - RQ = 0$$

Solving for $\check{P}$ and $\hat{P}$ gives us

$$\check{P} = \frac{Q \pm \sqrt{Q^2 + 4QR}}{2} \qquad \hat{P} = \frac{-Q \pm \sqrt{Q^2 + 4QR}}{2}$$

Note that $\check{P}$ and $\hat{P}$ must be positive def. (positive) which is only ~~be~~ true for 1 of the roots of $\check{P}$ and $\hat{P}$ respectively

$$\check{P} = \frac{Q \pm Q\sqrt{1 + \frac{4R}{Q}}}{2} = \frac{Q\left(1 \pm \sqrt{1 + \frac{4R}{Q}}\right)}{2}$$

Since Q and R are pos def $\check{P} = \frac{Q}{2}\left(1 + \sqrt{1 + 4\frac{R}{Q}}\right)$ is the only pos root.

$$\hat{P} = \frac{Q\left(-1 \pm \sqrt{1 + \frac{4R}{Q}}\right)}{2}$$

Since Q and R are pos def, $\hat{P} = \frac{Q}{2}\left(-1 + \sqrt{1 + \frac{4R}{Q}}\right)$ is the only positive root.

3.6.6 let $B = \begin{bmatrix} 1 & & & \\ A & 1 & & \\ A^2 & A & 1 & \\ \vdots & & & \ddots \\ A^k & A^{k-1} & \cdots & A \; 1 \end{bmatrix}$ Show $B^{-1} = \begin{bmatrix} 1 & & & \\ -A & 1 & & \\ & -A & \ddots & \\ & & -A & 1 \end{bmatrix}$

$$BB^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 & \cdots \end{bmatrix} = \overset{\text{identity matrix}}{I_{n \times n}}$$

Similarly $B^{-1}B = I$ $\therefore$ B^{-1} is indeed $\begin{bmatrix} 1 & & \\ -A & \ddots & \\ & -A & 1 \end{bmatrix}$ ▨

3.6.7 To solve for L : $O(N(K+1))$

To solve for L^{-1} : $O(N^2(K+1)^2)$

L^{-1} will be a lower triangle but not necessarilly sparse so no ~~optim~~ efficient technique that can be used.

$\therefore$ Solving $\hat{P}$ takes $O(N(K+1) + N^2(K+1)^2) = O(N^2(K+1)^2)$ ▨