Barfoot Prob 7.5.1

```
C = sym('C%d%d', [3, 3])
u = sym('u%d', [3, 1])
uHat = vechat(u)
```

```
C =

[ C11, C12, C13]
[ C21, C22, C23]
[ C31, C32, C33]


u =

 u1
 u2
 u3
```

$\mathbf{A} = (\mathbf{Cu})^\wedge$

```
Cu = C*u;
A = vechat(Cu)
```

```
A =

[                          0, - C31*u1 - C32*u2 - C33*u3,   C21*u1 +
 C22*u2 + C23*u3]
[   C31*u1 + C32*u2 + C33*u3,                          0, - C11*u1 -
 C12*u2 - C13*u3]
[ - C21*u1 - C22*u2 - C23*u3,   C11*u1 + C12*u2 + C13*u3,
            0]
```

$\mathbf{B} = \mathbf{C}\mathbf{u}^\wedge\mathbf{C}$

```
assume(C, 'real')
C*uHat*C'
```

```
ans =

[ C13*(C11*u2 - C12*u1) - C12*(C11*u3 - C13*u1) + C11*(C12*u3 -
 C13*u2), C23*(C11*u2 - C12*u1) - C22*(C11*u3 - C13*u1) + C21*(C12*u3
 - C13*u2), C33*(C11*u2 - C12*u1) - C32*(C11*u3 - C13*u1) +
 C31*(C12*u3 - C13*u2)]
[ C13*(C21*u2 - C22*u1) - C12*(C21*u3 - C23*u1) + C11*(C22*u3 -
 C23*u2), C23*(C21*u2 - C22*u1) - C22*(C21*u3 - C23*u1) + C21*(C22*u3
 - C23*u2), C33*(C21*u2 - C22*u1) - C32*(C21*u3 - C23*u1) +
 C31*(C22*u3 - C23*u2)]
```

```
[ C13*(C31*u2 - C32*u1) - C12*(C31*u3 - C33*u1) + C11*(C32*u3 -
 C33*u2), C23*(C31*u2 - C32*u1) - C22*(C31*u3 - C33*u1) + C21*(C32*u3
 - C33*u2), C33*(C31*u2 - C32*u1) - C32*(C31*u3 - C33*u1) +
 C31*(C32*u3 - C33*u2)]
```

Simplified $\mathbf{B} = \mathbf{Cu}^\wedge\mathbf{C}$

Use the cross product properties of C's basis to simplify the equation See Hughes' Spacecraft Attitude Dynamics Chapter 2.1

```
assumeAlso(C(:,1) == cross(C(:,2), C(:,3)))
assumeAlso(C(:,2) == cross(C(:,3), C(:,1)))
assumeAlso(C(:,3) == cross(C(:,1), C(:,2)))
B = simplify(C*uHat*C')
```

```
B =

[                          0, - C31*u1 - C32*u2 - C33*u3,   C21*u1 +
 C22*u2 + C23*u3]
[   C31*u1 + C32*u2 + C33*u3,                          0, - C11*u1 -
 C12*u2 - C13*u3]
[ - C21*u1 - C22*u2 - C23*u3,   C11*u1 + C12*u2 + C13*u3,
            0]
```

Comparing $\mathbf{A}$ and $\mathbf{B}$

```
isequal(A, B)
```

```
ans =

  logical

   1
```

Miscellaneous

```
function R = vechat(phi)
    R = [0 -phi(3) phi(2); phi(3) 0 -phi(1); -phi(2) phi(1) 0];
end
```

```
uHat =

[   0, -u3,  u2]
[  u3,   0, -u1]
[ -u2,  u1,   0]
```

7.5.2 Prove $(Cu)^\wedge = (2\cos\phi + 1)\, u^\wedge - u^\wedge C - C^T u^\wedge$

Note of the 2 properties (see pg 252 of barfoot ser.pdf)

- $(Wu)^\wedge = u^\wedge(\mathrm{tr}(W)\, 1 - W) - W^T u^\wedge$
- $\mathrm{tr}(C) = 2\cos\phi + 1$

$$(Cu^\wedge) = u^\wedge(\mathrm{tr}(C)\, 1 - C) - C^T u^\wedge$$
$$= (2\cos\phi + 1)\, u^\wedge - u^\wedge C - C^T u^\wedge \quad \blacksquare$$

7.5.3 Prove $\exp((Cu)^\wedge) = C \exp(u^\wedge)\, C^T$

By def. $\exp((Cu)^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (Cu)^{\wedge n}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (Cu^\wedge C^T)^n \quad \text{from prob 7.5.1}$$

Notice that $(Cu^\wedge C^T)^2 = Cu^\wedge C^T C u^\wedge C^T = C(u^\wedge)^2 C^T$

$$(Cu^\wedge C^T)^n = Cu^\wedge \cancel{C^T C} u^\wedge C^T \ldots Cu^\wedge C^T = C(u^\wedge)^n C^T$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} C (u^\wedge)^n C^T = C \sum_{n=0}^{\infty} \frac{1}{n!} (u^\wedge)^n C^T$$

$$\exp((Cu)^\wedge) = C \exp(u^\wedge)\, C^T \quad \blacksquare$$

7.5.4 $(\mathcal{T}x)^\wedge \equiv T x^\wedge T^{-1}$

Let $\mathcal{T} = \begin{bmatrix} C & r^\wedge C \\ 0 & C \end{bmatrix}$, $x = \begin{bmatrix} \rho \\ \phi \end{bmatrix}$, $T = \begin{bmatrix} C & r \\ 0 & 1 \end{bmatrix}$, $T^{-1} = \begin{bmatrix} C^T & -C^T r \\ 0 & 1 \end{bmatrix}$

① $(\mathcal{T}x)^\wedge = \left(\begin{bmatrix} C & r^\wedge C \\ 0 & C \end{bmatrix} \begin{bmatrix} \rho \\ \phi \end{bmatrix}\right)^\wedge = \begin{bmatrix} C\rho + r^\wedge C\phi \\ C\phi \end{bmatrix}^\wedge = \begin{bmatrix} (C\phi)^\wedge & C\rho + r^\wedge C\phi \\ 0 & 0 \end{bmatrix}$

② $T x^\wedge T^{-1} = \begin{bmatrix} C & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi^\wedge & \rho \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^T & -C^T r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} C\phi^\wedge & C\rho \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^T & -C^T r \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} C\phi^\wedge C^T & -C\phi^\wedge C^T r + C\rho \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (C\phi)^\wedge & -(C\phi)^\wedge r + C\rho \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (C\phi)^\wedge & r^\wedge (C\phi) + C\rho \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (C\phi)^\wedge & r^\wedge C\phi + C\rho \\ 0 & 0 \end{bmatrix}$$

Notice that ① ≡ ② $\blacksquare$

7.5.~~4~~5 Prove $\exp\left((Tx)^\wedge\right) = T\exp(x^\wedge)T^{-1}$

Solution is similar to 7.5.3

$$\exp\left((Tx)^\wedge\right) = \sum_{n=0}^{\infty}\frac{1}{n!}(Tx)^{\wedge n}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(T\,\cancel{\exp}\,x^\wedge T^{-1}\right)^n \quad \text{from p 7.5.4}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}T(x^\wedge)^n T^{-1} \quad \text{notice } Tx^\wedge \cancel{T^{-1}}\cancel{T}x^\wedge\cancel{T^{-1}}\cancel{T}x^\wedge T^{-1}\ldots Tx^\wedge T^{-1}$$
$$= T\sum_{n=0}^{\infty}\frac{1}{n!}(x^\wedge)^n T^{-1}$$
$$= T\exp(x^\wedge)T^{-1}$$

7.5.6 * skip for now * July 29, 2018

7.5.7 Prove $x^\wedge p \equiv p^\odot x$

Let $x^\wedge = \begin{bmatrix}\rho\\ \phi\end{bmatrix}^\wedge = \begin{bmatrix}\phi^\wedge & \rho\\ 0 & 0\end{bmatrix}$ 4x4 matrix

$p^\odot = \begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}^\odot = \begin{bmatrix}\eta\mathbf{1} & -\varepsilon^\wedge\\ 0 & 0\end{bmatrix}$ 4x6 matrix

Left side ①: $x^\wedge p = \begin{bmatrix}\phi^\wedge & \rho\\ 0 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix} = \begin{bmatrix}\phi^\wedge\varepsilon + \rho\eta\\ 0\end{bmatrix}$

Right side ②: $p^\odot x = \begin{bmatrix}\eta & -\varepsilon^\wedge\\ 0 & 0\end{bmatrix}\begin{bmatrix}\rho\\ \phi\end{bmatrix} = \begin{bmatrix}\rho\eta - \varepsilon^\wedge\phi\\ 0\end{bmatrix} = \begin{bmatrix}\phi^\wedge\varepsilon + \rho\eta\\ 0\end{bmatrix}$

① = ② $\therefore x^\wedge p \equiv p^\odot x$

7.5.8 Prove $p^T x^\wedge \equiv x^T p^\circledcirc$ let $p^\circledcirc = \begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}^\circledcirc = \begin{bmatrix}0 & \varepsilon\\ -\varepsilon^\wedge & 0\end{bmatrix}$

Left side ①: $\underset{p^T x^\wedge}{}$ $\begin{bmatrix}\varepsilon^T & \eta\end{bmatrix}\begin{bmatrix}\phi^\wedge & \rho\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\varepsilon^T\phi^\wedge & \varepsilon^T\rho\end{bmatrix}$

Right side ② $x^T p^\circledcirc = \begin{bmatrix}\rho^T & \phi^T\end{bmatrix}\begin{bmatrix}0 & \varepsilon\\ -\varepsilon^\wedge & 0\end{bmatrix} = \begin{bmatrix}-\phi^T\varepsilon^\wedge & \underbrace{\rho^T\varepsilon}_{\text{scalar}}\end{bmatrix}$

$= \begin{bmatrix}\varepsilon^T\phi^\wedge & \varepsilon^T\rho\end{bmatrix}$ Note $-\phi^T\varepsilon^\wedge = (\varepsilon^\wedge\phi)^T = (-\phi^\wedge\varepsilon)^T = \varepsilon^T\phi^\wedge$

① = ② $\therefore p^T x^\wedge \equiv x^T p^\circledcirc$

7.5.9 Prove $\int_0^1 \alpha^n (1-\alpha)^m \, d\alpha = \frac{n!\,m!}{(n+m+1)!}$

i.b.p. $\int \frac{du}{dt} v = uv - \int u \frac{dv}{dt}$

$$\int_0^1 \alpha^n (1-\alpha)^m \, d\alpha = \frac{\alpha^{n+1}(1-\alpha)^m}{n+1} + \frac{m}{n+1}\int \alpha^{n+1}(1-\alpha)^{m-1} d\alpha$$

let $u = \frac{\alpha^{n+1}}{n+1}$ $v = (1-\alpha)^m$

$du = \alpha^n d\alpha$ $dv = -m(1-\alpha)^{m-1} d\alpha$

$$= \frac{\alpha^{n+1}(1-\alpha)^m}{n+1} + \frac{m}{(n+1)(n+2)}\alpha^{n+2}(1-\alpha)^{m-1} + \frac{m(m-1)}{(n+1)(n+2)}\int \alpha^{n+2}(1-\alpha)^{m-2} d\alpha$$

let $u = \frac{\alpha^{n+2}}{n+2}$ $v = (1-\alpha)^{m-1}$

$du = \alpha^{n+1} d\alpha$ $dv = -(m-1)(1-\alpha)^{m-2} d\alpha$

continuing this i.b.p. expansion will give us

$$= \frac{\alpha^{n+1}(1-\alpha)^m}{n+1} + \frac{m}{(n+1)(n+2)}\alpha^{n+2}(1-\alpha)^{m-1} + \dots + \frac{(m)(m-1)\dots(2)}{(n+1)\dots(n+m)}\alpha^{n+m}(1-\alpha) + \frac{m!}{(n+1)\dots(n+m)}\int \alpha^{n+m} d\alpha$$

evaluate at $\alpha = 1$ minus $\alpha = 0$, all $\alpha^n(1-\alpha)^m$ terms become zero

$$= \frac{m!}{(n+1)\dots(n+m)}\int_0^1 \alpha^{n+m} d\alpha = \frac{n!\,m!}{n!\,(n+1)\dots(n+m)}\left(\frac{\alpha^{n+m+1}}{n+m+1}\Bigg|_{\alpha=1} - \frac{\alpha^{n+m+1}}{n+m+1}\Bigg|_{\alpha=0} \right)$$

$$= \frac{n!\,m!}{(n+m+1)!} \blacksquare$$

7.5.10 skip for now July 29, 2018

7.5.11 Show $(\mathcal{T}p)^{\odot} \equiv \mathcal{T}p^{\odot}\mathcal{T}^{-1}$

Left side ①: $(\mathcal{T}p)^{\odot} = \left(\begin{bmatrix} C & J\rho \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}\right)^{\odot} = \begin{bmatrix} C\varepsilon + J\rho\eta \\ \eta \end{bmatrix}^{\odot}$

$= \begin{bmatrix} \eta 1 & -(C\varepsilon + J\rho\eta)^{\wedge} \\ 0 & 0 \end{bmatrix}$

Right side ②: $\mathcal{T}p^{\odot}\mathcal{T}^{-1} = \begin{bmatrix} C & J\rho \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \eta 1 & -\varepsilon^{\wedge} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C^{T} & -C^{T}(J\rho)^{\wedge} \\ 0 & C^{T} \end{bmatrix}$

$= \begin{bmatrix} \eta C & -C\varepsilon^{\wedge} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C^{T} & -C^{T}(J\rho)^{\wedge} \\ 0 & C^{T} \end{bmatrix}$ $C\varepsilon^{\wedge}C^{T} = (C\varepsilon)^{\wedge}$

$= \begin{bmatrix} \eta 1 & -\eta(J\rho)^{\wedge} - C\varepsilon^{\wedge}C^{T} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \eta 1 & -(\eta(J\rho)^{\wedge} + (C\varepsilon)^{\wedge}) \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} \eta 1 & -(\eta J\rho + C\varepsilon)^{\wedge} \\ 0 & 0 \end{bmatrix}$ note $^{\wedge}$ is distributive

Since ① = ②, $(\mathcal{T}p)^{\odot} \equiv \mathcal{T}p^{\odot}\mathcal{T}^{-1}$ ∎

7.5.12 Show $(\mathcal{T}p)^{\odot^{T}}(\mathcal{T}p)^{\odot} \equiv \mathcal{T}^{-T}p^{\odot^{T}}p^{\odot}\mathcal{T}^{-1}$

$(\mathcal{T}p)^{\odot^{T}}(\mathcal{T}p)^{\odot} = (\mathcal{T}p^{\odot}\mathcal{T}^{-1})^{T}(\mathcal{T}p^{\odot}\mathcal{T}^{-1})$ from P7.5.11

$= \mathcal{T}^{-T}p^{\odot^{T}}\mathcal{T}^{T}\mathcal{T}p^{\odot}\mathcal{T}^{-1}$ since $p^{\odot^{T}}\mathcal{T}^{T}\mathcal{T}p^{\odot} = p^{\odot^{T}}p^{\odot}$

$= \mathcal{T}^{-T}p^{\odot^{T}}p^{\odot}\mathcal{T}^{-1}$ ∎ from proof below...

$p^{\odot^{T}}\mathcal{T}^{T}\mathcal{T}p^{\odot} = \begin{bmatrix} A^{T} & 0 \\ B^{T} & 0 \end{bmatrix}\begin{bmatrix} C^{T} & 0 \\ (J\rho)^{T} & 1 \end{bmatrix}\begin{bmatrix} C & J\rho \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} A^{T}C^{T} & 0 \\ B^{T}C^{T} & 0 \end{bmatrix}\begin{bmatrix} CA & CB \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^{T}C^{T}CA & A^{T}C^{T}CB \\ B^{T}C^{T}CA & B^{T}C^{T}CB \end{bmatrix} = \begin{bmatrix} A^{T}A & A^{T}B \\ B^{T}A & B^{T}B \end{bmatrix}$ //

$p^{\odot^{T}}p^{\odot} = \begin{bmatrix} A^{T} & 0 \\ B^{T} & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^{T}A & A^{T}B \\ B^{T}A & B^{T}B \end{bmatrix}$ //

$\therefore p^{\odot^{T}}\mathcal{T}^{T}\mathcal{T}p^{\odot} = p^{\odot^{T}}p^{\odot}$

partial sol'n... I still don't know how to handle $(\dot{r}^{\wedge}C)$

7.5.13 Note $\dot{C} = w^{\wedge}C$ from eq. 7.198 ①

$\dot{T} = \begin{bmatrix} \dot{C} & \dot{r} \\ 0 & 0 \end{bmatrix} = \bar{w}^{\wedge}T = \begin{bmatrix} v \\ w \end{bmatrix}^{\wedge}\begin{bmatrix} C & r \\ 0 & \cancel{1} \end{bmatrix} = \begin{bmatrix} w^{\wedge} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C & r \\ 0 & \cancel{1} \end{bmatrix} = \begin{bmatrix} w^{\wedge}C & w^{\wedge}\cancel{rv} \\ 0 & 0 \end{bmatrix}$

$\dot{\mathcal{T}} = \begin{bmatrix} \dot{C} & (\dot{r}^{\wedge}C) \\ 0 & \dot{C} \end{bmatrix} = \begin{bmatrix} w^{\wedge}C & \cancel{\cdots} \\ 0 & w^{\wedge}C \end{bmatrix} = \begin{bmatrix} w^{\wedge}C & \cancel{\cdots} \\ 0 & w^{\wedge}C \end{bmatrix}$

from ① since $r = w^{\wedge}r$

$\bar{w}^{\lambda}\mathcal{T} = \begin{bmatrix} w^{\wedge} & v^{\wedge} \\ 0 & w^{\wedge} \end{bmatrix}\begin{bmatrix} C & r^{\wedge}C \\ 0 & C \end{bmatrix} = \begin{bmatrix} w^{\wedge}C & w^{\wedge}r^{\wedge}C + v^{\wedge}C \\ 0 & w^{\wedge}C \end{bmatrix}$

7.5.14 We can work w/ a modified version of the homogenous point represation by

$$T p = Ad^{-1}\left(Ad(T)\, Ad(p)\right) \qquad \text{where } p = \begin{bmatrix} c \\ 1 \end{bmatrix}$$

$$= Ad^{-1}\left(\begin{bmatrix} C & (J\rho)^\wedge C \\ 0 & C \end{bmatrix} \begin{bmatrix} c^\wedge \\ 1 \end{bmatrix} \right)$$

$$= Ad^{-1}\left(\begin{bmatrix} Cc^\wedge + (J\rho)^\wedge C \\ C \end{bmatrix} \right) = \begin{bmatrix} \left(\left(Cc^\wedge + (J\rho)^\wedge C\right) C^T\right)^\vee \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(Cc^\wedge C^T + (J\rho)^\wedge\right)^\vee \\ 1 \end{bmatrix} = \begin{bmatrix} \left((Cc) + J\rho\right)^{\wedge\vee} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} Cc + J\rho \\ 1 \end{bmatrix}$$

Note: $T p = \begin{bmatrix} C & J\rho \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ 1 \end{bmatrix} = \begin{bmatrix} Cc + J\rho \\ 1 \end{bmatrix}$