Dynamics: Theory and Applications - Kane

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November 2017

1 Differentiation of Vectors

1.1 Vector Functions

When the magnitude or direction of $\boldsymbol{v}$ is dependent on q in frame A, $\boldsymbol{v}$ is called a function of q in A. Otherwise, we say $\boldsymbol{v}$ is independent of q in A.

1.2 Several Reference Frames

$\boldsymbol{v}$ may be a function of q in frame A but not in frame B.

1.3 Scalar Functions

Given a reference frame A (3D)

$$\boldsymbol{v} = v_1\boldsymbol{a_1} + v_2\boldsymbol{a_2} + v_3\boldsymbol{a_3} \tag{1.1}$$

$v_i\boldsymbol{a_i}$ is called the $\boldsymbol{a_i}$ component of $\boldsymbol{v}$, and v_i is called the $\boldsymbol{a_i}$ measure number of $\boldsymbol{v}$.

When $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$ are mutually perpendicular, $v_i = \boldsymbol{v}\cdot\boldsymbol{a_i}$.

$$\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{a_1}\boldsymbol{a_1} + \boldsymbol{v}\cdot\boldsymbol{a_2}\boldsymbol{a_2} + \boldsymbol{v}\cdot\boldsymbol{a_3}\boldsymbol{a_3} \tag{1.2}$$

1.4 First Derivatives

If $\boldsymbol{v}$ is a vector function of n scalar variables $q_1, \ldots, q_n$ in frame A,

$$\frac{{}^A\delta\boldsymbol{v}}{\delta q_r} \triangleq \sum_{i=1}^{3} \frac{\delta v_i}{\delta q_r}\boldsymbol{a_i} \qquad ,(r = 1, \ldots, n) \tag{1.3}$$

1.5 Representation of Derivatives

Derivative in frame A is not necesarrily equal to derivative in frame B.

1.6 Notation for Derivatives

No mention of reference implies (i) any frame can be used or (ii) all the subsequent eq. are in the same frame.

1.7 Differentiation of Sums and Products

$$\frac{\delta}{\delta q_r}\boldsymbol{v} = \sum_{i=1}^{N} \frac{\delta v_i}{\delta q_r} \text{ for } (r = 1, \ldots, n) \tag{1.4}$$

$$\frac{\delta}{\delta q_r}(s\boldsymbol{v}) = \frac{\delta s}{\delta q_r}\boldsymbol{v} + s\frac{\delta \boldsymbol{v}}{\delta q_r} \tag{1.5}$$

$$\frac{\delta}{\delta q_r}(\boldsymbol{v}\cdot\boldsymbol{w}) = \frac{\delta \boldsymbol{v}}{\delta q_r}\cdot\boldsymbol{w} + \boldsymbol{v}\cdot\frac{\delta \boldsymbol{w}}{\delta q_r} \tag{1.6}$$

$$\frac{\delta}{\delta q_r}(\boldsymbol{v}\times\boldsymbol{w}) = \frac{\delta \boldsymbol{v}}{\delta q_r}\times\boldsymbol{w} + \boldsymbol{v}\times\frac{\delta \boldsymbol{w}}{\delta q_r} \tag{1.7}$$

$$\text{If } P = F_1F_2\ldots F_N, \text{ in general,} \tag{1.8}$$

$$\frac{\delta P}{\delta q_r} = \frac{\delta F_1}{\delta q_r}F_2\ldots F_N + \cdots + F_1F_2\ldots\frac{\delta F_N}{\delta q_r} \tag{1.9}$$

1.8 Second Derivatives

At different reference frame, order is important. At similar reference frame, order is not important.

$$\frac{{}^B\delta}{\delta q_s}\frac{{}^A\delta}{\delta q_r} \neq \frac{{}^A\delta}{\delta q_s}\frac{{}^B\delta}{\delta q_r} \qquad (r, s = 1, \ldots, n) \tag{1.10}$$

$$\frac{\delta}{\delta q_s}\frac{\delta}{\delta q_r} = \frac{\delta}{\delta q_s}\frac{\delta}{\delta q_r} \qquad (r, s = 1, \ldots, n) \tag{1.11}$$

1.9 Total and Partial Derivatives

$$\frac{{}^A d\boldsymbol{v}}{dt} = \sum_{r=1}^{n} \frac{{}^A\delta\boldsymbol{v}}{\delta q_r} + \frac{{}^A\delta\boldsymbol{v}}{\delta t} \tag{1.12}$$

$$\frac{d}{dt}\frac{\delta\boldsymbol{v}}{\delta q_r} = \frac{\delta}{\delta q_r}\frac{d\boldsymbol{v}}{dt} \tag{1.13}$$

2 Kinematics

- 1-5 rotational motion of a rigid body.
- 6-8 translational motion of a point
- 9-13 constraints
- 14-15 partial linear and angular velocity

2.1 Angular Velocity

Though abstract, *angular velocity*'s definition provides a sound basis for the derivation of theorems used to solve problems.

Let $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, $\boldsymbol{b_3}$ form a right handed set of mutually perpendicular unit vectors fixed in a rigid body B moving in a reference frame A. The angular velocity of B in A is denoted by

$${}^A\boldsymbol{\omega}^B \triangleq \boldsymbol{b_1}\cdot\frac{{}^A d\mathbf{b_2}}{dt}\cdot\boldsymbol{b_3} + \boldsymbol{b_2}\cdot\frac{{}^A d\mathbf{b_3}}{dt}\cdot\boldsymbol{b_1} + \boldsymbol{b_3}\cdot\frac{{}^A d\mathbf{b_1}}{dt}\cdot\boldsymbol{b_2} \tag{2.1}$$

$$\frac{{}^A d\beta}{dt} = {}^A\boldsymbol{\omega}^B \times \boldsymbol{\beta} \tag{2.2}$$

$$\beta = \text{ any vector fixed in ref } B \tag{2.3}$$

2.2 Simple Angular Velocity

When a rigid body B move in frame A in such a way that a unit vector $\boldsymbol{k}$ is independent of t in both A and B, then B is said to have a simple angular velocity in A throughout this time interval. Note that B need not be mounted in A for B to have a simple angular velocity in A.

$${}^A\boldsymbol{\omega}^B = \omega\boldsymbol{k} \tag{2.4}$$

$$\omega \triangleq \dot{\theta} \tag{2.5}$$

2.3 Differentiation in Two Reference Frames

$$\frac{{}^A d\mathbf{v}}{dt} = \frac{{}^B d\mathbf{v}}{dt} + {}^A\boldsymbol{\omega}^B \times \boldsymbol{v} \tag{2.6}$$

2.4 Auxiliary Reference Frames

Addition theorem for angular velocities.

$$^A\boldsymbol{\omega}^B = {}^A\boldsymbol{\omega}^{A_1} + {}^{A_1}\boldsymbol{\omega}^{A_2} + \cdots + {}^{A_n}\boldsymbol{\omega}^B \tag{2.7}$$

Specially useful if each $\boldsymbol{\omega}$ are simple angular velocity. This has no angular acceleration counterpart.

2.5 Angular Acceleration

$$^A\boldsymbol{\alpha}^B \triangleq \frac{^A d\,{}^A\boldsymbol{\omega}^{\mathbf{B}}}{dt} = \frac{^B d\,{}^A\boldsymbol{\omega}^{\mathbf{B}}}{dt} \tag{2.8}$$

There is no angular acceleration counterpart for the addition theorem.

When B has a simple angular velocity in A, we have the ff. where α is called the scalar angular acceleration.

$$^A\boldsymbol{\alpha}^B = \alpha\boldsymbol{k} \tag{2.9}$$

$$\alpha = \frac{d\omega}{dt} \tag{2.10}$$

2.6 Velocity and Acceleration

$$^A\boldsymbol{v}^P \triangleq \frac{^A d\mathbf{p}}{dt} \tag{2.11}$$

$$^A\boldsymbol{a}^P \triangleq \frac{^A d\,{}^A\boldsymbol{v}^{\mathbf{P}}}{dt} \tag{2.12}$$

2.7 Two points fixed on a Rigid Body

If P and Q are two points fixed on a rigid body B having an angular velocity $^A\boldsymbol{\omega}^B$ in A,

$$^A\boldsymbol{v}^P = {}^A\boldsymbol{v}^Q + {}^A\boldsymbol{\omega}^B \times \boldsymbol{r} \tag{2.13}$$

$$^A\boldsymbol{a}^P = {}^A\boldsymbol{a}^Q + {}^A\boldsymbol{\omega}^B \times ({}^A\boldsymbol{\omega}^B \times \boldsymbol{r}) + {}^A\boldsymbol{\alpha}^B \times \boldsymbol{r} \tag{2.14}$$

$$\boldsymbol{r} = \text{ vector from point Q to P} \tag{2.15}$$

2.8 One point moving on a Rigid Body

If a point P is moving on a rigid body B while B is moving in a reference frame A, then we have the ff. where $2\,{}^A\boldsymbol{\omega}^B \times {}^B\boldsymbol{v}^P$ is referred as the Coriolis acceleration.

$$^A\boldsymbol{v}^P = {}^A\boldsymbol{v}^{\bar{B}} + {}^B\boldsymbol{v}^P \tag{2.16}$$

$$^A\boldsymbol{a}^P = {}^A\boldsymbol{a}^{\bar{B}} + {}^B\boldsymbol{a}^P + 2\,{}^A\boldsymbol{\omega}^B \times {}^B\boldsymbol{v}^P \tag{2.17}$$

2.9 Configuration Constraints

If subject S is affected by other bodies (e.g., contact), it is subject to configuration constraints.

1. Holonomic constraint equations = equations expressing restrictions that is of the form $f(x_1, y_1, z_1, \ldots, x_v, y_v, z_v, t) = 0$.
2. Rheonomic = holomic constraint equation is DEPENDENT on time t.
3. Scleronomic = holomic constraint equation is NOT DEPENDENT on time t.

2.10 Generalized Coordinates

- When a set S has v points subject to M Holonomic constraint equations, it has $n = 3v - M$ independent equations.
- One can express $x_i, y_i, z_i (i = 1, \ldots, v)$ as $q_1(t), \ldots, q_n(t)$. The values of $q_1(t), \ldots, q_n(t)$ are called the generalized coordinates for S in A.

2.11 Number of Generalized Coordinates

2.12 Generalized Speeds

Kinematical differential equations for S in A. Generalized speeds can be time-derivatives of the generalized coordinates and time, but this is not always the case.

$$u_r \triangleq \sum_{s=1}^{n} Y_{rs}\dot{q}_s + Z_r \qquad (r = 1, \ldots, n) \tag{2.18}$$

where Y_{rs} and Z_r are functions of $q_1, \ldots, q_n$ and t.

2.13 Motion Constraints

- Nonholonomic constraint equations = equations expressing motion constraints.
- If S is not subject to motion constraints, then S is said to be a simple holonomic system possessing n degrees of freedom in A.
- If S is subject to motion constraints, then S is said to be a nonholonomic system.

$$u_r \triangleq \sum_{s=1}^{p} A_{rs} \cdot u_s + B_r \qquad (r = p+1, \ldots, n) \tag{2.19}$$

$$p \triangleq n - m \tag{2.20}$$

where A_{rs} and B_r are functions of $q_1, \ldots, q_n$ and t.

2.14 Partial Angular Velocities, Partial Velocities

The angular velocity, $\boldsymbol{\omega}$, in A of rigid body B and the velocity, $\boldsymbol{v}$, in A of particle P belonging to S, can be unique expressed as

$$\boldsymbol{\omega} = \sum_{r=1}^{n} \boldsymbol{\omega}_r u_r + \boldsymbol{\omega}_t \tag{2.21}$$

$$\boldsymbol{v} = \sum_{r=1}^{n} \boldsymbol{v}_r u_r + \boldsymbol{v}_t \tag{2.22}$$

$$\boldsymbol{\omega} = \sum_{r=1}^{p} \tilde{\boldsymbol{\omega}}_r u_r + \tilde{\boldsymbol{\omega}}_t \tag{2.23}$$

$$\boldsymbol{v} = \sum_{r=1}^{p} \tilde{\boldsymbol{v}}_r u_r + \tilde{\boldsymbol{v}}_t \tag{2.24}$$

$$\boldsymbol{\omega_r} \triangleq \tag{2.25}$$

$$\boldsymbol{\omega_t} \triangleq \tag{2.26}$$

$$\boldsymbol{v_r} \triangleq \sum_{s=1}^{n} \frac{\delta\boldsymbol{p}}{\delta q_s} W_{sr},\ (r = 1, \ldots, n) \tag{2.27}$$

$$\boldsymbol{v_t} \triangleq \sum_{s=1}^{n} \frac{\delta\boldsymbol{p}}{\delta q_s} X_s + \frac{\delta\boldsymbol{p}}{\delta t} \tag{2.28}$$

$$\tilde{\boldsymbol{\omega}}_{\boldsymbol{r}} \triangleq \boldsymbol{\omega_r} + \sum_{s=p+1}^{n} \boldsymbol{\omega_s} A_{sr} \tag{2.29}$$

$$\tilde{\boldsymbol{\omega}}_{\boldsymbol{t}} \triangleq \boldsymbol{\omega_t} + \sum_{r=p+1}^{n} \boldsymbol{\omega_r} B_r \tag{2.30}$$

$$\tilde{\boldsymbol{v}}_{\boldsymbol{r}} \triangleq \boldsymbol{v_r} + \sum_{s=p+1}^{n} \boldsymbol{v_s} A_{sr} \tag{2.31}$$

$$\tilde{\boldsymbol{v}}_{\boldsymbol{t}} \triangleq \boldsymbol{v_t} + \sum_{r=p+1}^{n} \boldsymbol{v_r} B_r \tag{2.32}$$

where $\boldsymbol{\omega}_r$, $\boldsymbol{v}_r$, $\tilde{\boldsymbol{\omega}}_r$, $\tilde{\boldsymbol{v}}_r$ are the r^{th} partial holonomic angular velocity, holonomic velocity, nonholonomic angular velocity, nonholonomic velocity, respectively, and are functions of $q_1, \dots, q_n$; $\boldsymbol{\omega}_t, \boldsymbol{v}_t$, $\tilde{\boldsymbol{\omega}}_t$, $\tilde{\boldsymbol{v}}_t$ are functions of t.

2.15 Acceleration and Partial Velocities

$$\boldsymbol{v}_r \cdot \boldsymbol{a} = \frac{1}{2}\left(\frac{d}{dt}\frac{\delta \boldsymbol{v^2}}{\delta \dot{q}_r} - \frac{\delta \boldsymbol{v^2}}{\delta q_r}\right) \tag{2.33}$$

$$\boldsymbol{v}_r \cdot \boldsymbol{a} = \frac{1}{2}\sum_{s=1}^{n}\left(\frac{d}{dt}\frac{\delta \boldsymbol{v^2}}{\delta \dot{q}_s} - \frac{\delta \boldsymbol{v^2}}{\delta q_s}\right)W_{sr} \tag{2.34}$$

$$\tilde{\boldsymbol{v}}_r \cdot \boldsymbol{a} = \frac{1}{2}\left(\frac{d}{dt}\frac{\delta \boldsymbol{v^2}}{\delta \dot{q}_r} - \frac{\delta \boldsymbol{v^2}}{\delta q_r}\right) + \tag{2.35}$$

$$\frac{1}{2}\sum_{s=p+1}^{n}\left(\frac{d}{dt}\frac{\delta \boldsymbol{v^2}}{\delta \dot{q}_s} - \frac{\delta \boldsymbol{v^2}}{\delta q_s}\right)A_{sr} \tag{2.36}$$

$$\tilde{\boldsymbol{v}}_r \cdot \boldsymbol{a} = \frac{1}{2}\sum_{s=1}^{n}\left[\left(\frac{d}{dt}\frac{\delta \boldsymbol{v^2}}{\delta \dot{q}_s} - \frac{\delta \boldsymbol{v^2}}{\delta q_s}\right)\left(W_{sr} + \sum_{k=p+1}^{n} W_{sk}A_{kr}\right)\right] \tag{2.37}$$

3 Mass Distribution

3.1 Mass Center

Let S be a set of particles $P_1, \dots, P_v$ of masses $m_1, \dots, m_v$, $\boldsymbol{r_i}$ be the distance between the mass center S^* to $P_1, \dots, P_v$, $\boldsymbol{p}^*$ be the position vector from O to S^*.

$$\sum_{i=1}^{v} m_i \boldsymbol{r_i} = 0 \tag{3.1}$$

$$\boldsymbol{p^*} = \frac{\sum_{i=1}^{v} m_i \boldsymbol{p_i}}{\sum_{i=1}^{v} m_i} \tag{3.2}$$

3.2 Curves, Surfaces, and Solids

Let B^* be the mass center, ρ be the mass density, $d\tau$ be the length/area/volume of a differential element of figure F, $\boldsymbol{p}$ be the position vector from O to P, $\boldsymbol{p}^*$ be the position vector from O to B^*.

$$\int_F \rho \boldsymbol{r} d\tau = 0 \tag{3.3}$$

$$\boldsymbol{p^*} = \frac{\int_F \rho \boldsymbol{p} d\tau}{\int_F \rho d\tau} \tag{3.4}$$

3.3 Inertia Vector, Inertia Scalars

Let S be a set of particles $P_1, \dots, P_v$ of masses $m_1, \dots, m_v$, $\boldsymbol{p_i}$ be the position vector from a point O to P_i, $\boldsymbol{n_a}$ is a unit vector.

- $\boldsymbol{I_a}$ = inertia vector of S relative to O for $\boldsymbol{n_a}$
- I_{ab} = inertia scalar of S relative to O for $\boldsymbol{n_a}$ and $\boldsymbol{n_b}$
- I_a = moment of inertia of S with respect to line L_a, where L_a is the line passing through point O and parallel to $\boldsymbol{n_a}$. (I_{aa})

$$\boldsymbol{I_a} \triangleq \sum_{i=1}^{v} m_i \boldsymbol{p_i} \times (\boldsymbol{n_a} \times \boldsymbol{p_i}) \tag{3.5}$$

$$I_{ab} \triangleq \boldsymbol{I_a} \cdot \boldsymbol{n_b} = I_{ba} \tag{3.6}$$

$$= \sum_{i=1}^{v} m_i (\boldsymbol{p_i} \times \boldsymbol{n_a}) \cdot (\boldsymbol{p_i} \times \boldsymbol{n_b}) \tag{3.7}$$

$$I_a = \sum_{i=1}^{v} m_i (\boldsymbol{p_i} \times \boldsymbol{n_a})^2 \tag{3.8}$$

$$= \sum_{i=1}^{v} m_i l_i^2 = m k_a^2 \tag{3.9}$$

$$\boldsymbol{I_a} \triangleq \int_{\boldsymbol{F}} \rho \boldsymbol{p} \times (\boldsymbol{n_a} \times \boldsymbol{p}) d\tau \tag{3.10}$$

$$I_{ab} \triangleq \boldsymbol{I_a} \cdot \boldsymbol{n_b} = \int_{\boldsymbol{F}} \rho (\boldsymbol{p} \times \boldsymbol{n_a}) \cdot (\boldsymbol{p} \times \boldsymbol{n_b}) d\tau \tag{3.11}$$

$$I_a = \int_{\boldsymbol{F}} \rho l^2 d\tau \tag{3.12}$$

3.4 Mutually Perpendicular Unit Vectors

Given inertia vectors $\boldsymbol{I_1}, \boldsymbol{I_2}, \boldsymbol{I_3}$ of a body B relative to a point O for three mutually perpendicular unit vectors $\boldsymbol{n_1}, \boldsymbol{n_2}, \boldsymbol{n_3}$,

$$\boldsymbol{I_a} = \sum_{j=1}^{3} a_j \boldsymbol{I_j} \tag{3.13}$$

$$a_j \triangleq \boldsymbol{n_a} \cdot \boldsymbol{n_j} \text{ for } j = 1, 2, 3 \tag{3.14}$$

$$I_{ab} = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j I_{jk} b_k \tag{3.15}$$

$$b_k \triangleq \boldsymbol{n_b} \cdot \boldsymbol{n_k} \text{ for } k = 1, 2, 3 \tag{3.16}$$

3.5 Inertia Matrix, Inertia Dyadic

3.5.1 Inertia Matrix

Each inertia matrix is associated with a specific basis vector. Set S does not possess a unique inertia matrix relative to O.

$$I \triangleq \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \tag{3.17}$$

$$a \triangleq [a_1 \quad a_2 \quad a_3] \tag{3.18}$$

$$b \triangleq [b_1 \quad b_2 \quad b_3] \tag{3.19}$$

$$I_{ab} = a I b^T \tag{3.20}$$

3.5.2 Inertia Dyadic

Basis independent.

$$\boldsymbol{u} = \boldsymbol{w} \cdot \boldsymbol{ab} + \boldsymbol{w} \cdot \boldsymbol{cd} + \dots \tag{3.21}$$

$$\boldsymbol{v} = \boldsymbol{ab} \cdot \boldsymbol{w} + \boldsymbol{cd} \cdot \boldsymbol{w} + \dots \tag{3.22}$$

$$\boldsymbol{Q} \triangleq \boldsymbol{ab} + \boldsymbol{cd} + \dots \text{ dyadic} \tag{3.23}$$

$$\boldsymbol{u} = w \cdot \boldsymbol{Q} \text{ scalar premultiplication} \tag{3.24}$$

$$\boldsymbol{v} = \boldsymbol{Q} \cdot w \text{ scalar postmultiplication} \tag{3.25}$$

$$\boldsymbol{U} \triangleq \boldsymbol{a}_1\boldsymbol{a}_1 + \boldsymbol{a}_2\boldsymbol{a}_2 + \boldsymbol{a}_3\boldsymbol{a}_3 \tag{3.26}$$

$$\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{U} = \boldsymbol{U} \cdot \boldsymbol{v} \tag{3.27}$$

where $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are mutually perpendicular unit vectors.

$$\boldsymbol{I} \triangleq \sum_{i=1}^{v} m_i(\boldsymbol{U}\boldsymbol{p}_i^2 - \boldsymbol{p}_i\boldsymbol{p}_i) \tag{3.28}$$

$$\triangleq \int_{\boldsymbol{F}} \rho(\boldsymbol{U}\boldsymbol{p}^2 - \boldsymbol{p}\boldsymbol{p})d\tau \tag{3.29}$$

$$= \sum_{j=1}^{3} \boldsymbol{I_j}\boldsymbol{n_j} \tag{3.30}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3} I_{jk}\boldsymbol{n_j}\boldsymbol{n_k} \tag{3.31}$$

$$\boldsymbol{I_a} = \boldsymbol{n_a} \cdot \boldsymbol{I} \tag{3.32}$$

$$I_{ab} = \boldsymbol{n_a} \cdot \boldsymbol{I} \cdot \boldsymbol{n_b} \tag{3.33}$$

3.5.3 Angular Momentum

$${}^A\boldsymbol{H}^{S/O} \triangleq \sum_{i=1}^{v} m_i\boldsymbol{p_i} \times {}^A\boldsymbol{v}^{P_i} \tag{3.34}$$

$${}^A\boldsymbol{H}^{B/O} = \boldsymbol{I}^{B/O} \cdot {}^A\boldsymbol{\omega}^B \tag{3.35}$$

3.6 Parallel Axes Theorems

- Inertia dyadic $\boldsymbol{I}^{S/O}$ of a set S of v particles relative to a point O
- Central inertia dyadic $\boldsymbol{I}^{S/S*}$
- Central inertia scalars $I_{ab}^{S/S*}$ and $I_a^{S/S*}$

$$\boldsymbol{I}^{S/O} = \boldsymbol{I}^{S/S*} + \boldsymbol{I}^{S*/O} \text{ dyadic} \tag{3.36}$$

$$I^{S/O} = I^{S/S*} + I^{S*/O} \text{ inertia matrix} \tag{3.37}$$

$$\boldsymbol{I}_a^{S/O} = \boldsymbol{I}_a^{S/S*} + \boldsymbol{I}_a^{S*/O} \text{ inertia vector} \tag{3.38}$$

$$I_{ab}^{S/O} = I_{ab}^{S/S*} + I_{ab}^{S*/O} \text{ products of inertia} \tag{3.39}$$

$$I_a^{S/O} = I_a^{S/S*} + I_a^{S*/O} \text{ moments of inertia} \tag{3.40}$$

3.7 Evaluation of Inertia Scalars

1. discrete I_{ab}
   (a) by definition (eq. 3.7)
   (b) central inertia scalar + parallel axis
   (c) utilize inertia vector, matrix, or dyadic
2. continuous I_{ab}
   (a) use tables (Appendix I of kane dynamics)
   (b) assume uniform mass + dyadic
   (c) by definition is done as last resort

$$k = \sqrt{\frac{dI}{dM}} \text{ radius of gyration} \tag{3.41}$$

3.8 Principal Moments of Inertia

- principal axis of S for O: line L_z passing through O such that $\boldsymbol{n}_z$ is parallel to $\boldsymbol{I}_z$
- principal plane of S for O: plane P_z passing through O normal to $\boldsymbol{n}_z$
- principal moment of inertia of S for O: moment of inertia I_z with respect to L_z
- principal radius of gyration of S for O: radius of gyration of S with respect to L_z
- if point $O = S^*$, then you add 'central' to the name

$$\boldsymbol{I_z} = I_z\boldsymbol{n_z} \tag{3.42}$$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{3.43}$$

$$\tan(2\theta) = \frac{2I_{ab}}{I_a - I_b} \tag{3.44}$$

$$I_x, I_y = \frac{I_a + I_b}{2} \pm \left[\left(\frac{I_a - I_b}{2}\right) + I_{ab}^2\right]^{1/2} \tag{3.45}$$

3.9 Maximum and Minimum Moments of Inertia

4 Generalized Forces

4.1 Moment about a point, bound vectors, resultant

$$\boldsymbol{M} \triangleq \boldsymbol{p} \times \boldsymbol{v} \text{ moment of v about point P} \tag{4.1}$$

$$\boldsymbol{R} \triangleq \sum_{i=1}^{v} \boldsymbol{v_i} \tag{4.2}$$

$$\boldsymbol{M^{S/P}} = \boldsymbol{M^{S/Q}} + \boldsymbol{r^{PQ}} \times \boldsymbol{R} \tag{4.3}$$

1. $\boldsymbol{p}$ = position vector from point P to any point on line L
2. $\boldsymbol{M^{S/P}}$ = sum of v_i moments about point P = moment of S about P

4.2 Couples, Torque

- couple = set of bound vectors whole resultant is zero.
- simple couple = only 2 vectors in set.
- a couple has the same moment about all points.

4.3 Equivalence, Replacement

- 2 sets of bound vectors are "equivalent" when they have equal resultants and equal moments about one point. either set is called a "replacement" of the other.
- If 2 sets are equivalent, they have equal moments about every point.
- A set S can replaced by a set S' consisting of $\boldsymbol{T}$ equal to the moment of S about P and $\boldsymbol{v}$ equal to the resultant of S.

4.4 Generalized Active Forces

If $u_1, \ldots u_n$ are generalized speeds for a simple nonholonomic system S possessing p degrees of greedom in a reference frame A,

$$\tilde{F}_r \triangleq \sum_{i=1}^{v} \tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{P_i}} \cdot \boldsymbol{R_i} \qquad (r = 1, \ldots, p) \text{ nonholonomic} \tag{4.4}$$

$$F_r \triangleq \sum_{i=1}^{v} \boldsymbol{v}_{\boldsymbol{r}}^{\boldsymbol{P_i}} \cdot \boldsymbol{R_i} \qquad (r = 1, \ldots, n) \text{ holonomic} \tag{4.5}$$

$$\tilde{F}_r = F_r + \sum_{s=p+1}^{n} F_s A_{sr} \tag{4.6}$$

where v is the number of particle in set S, P_i is a typical particle of S, $\tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{P_i}}$ and $\boldsymbol{v}_{\boldsymbol{r}}^{\boldsymbol{P_i}}$ are the nonholonomic/holonomic partial velocity of P_i in A, and $\boldsymbol{R_i}$ is the resultant of all contact forces (e.g., friction) and distance forces (e.g., gravity) acting on P_i.

4.5 Noncontributing Forces

Contribution to $\tilde{F}_r$ of:

- All contact forces exerted on particles of S across smooth surfaces of rigid bodies vanishes.
- If B is a rigid body belonging to S, all contact and distance forces exerted by all particles of B on each other is equal to zero.
- When B rolls without slipping on a rigid body B'
  - all contact forces exerted on B by B' is equal to zero if B' is not part of S.
  - all contact forces exerted by B and B' on each other equal to zero if B' is part of S.

4.6 Forces Acting on a Rigid Body

If B is a rigid body belonging to a nonholonomic system S possessing p DoF in reference frame A, and a set of contact/distance forces acting on B is equivalent to a couple of torque $\boldsymbol{T}$ and force R on point Q of B, then $(\tilde{F}_r)_B$ the contribution of this set of forces to $\tilde{F}_r$ is

$$(\tilde{F}_r)_B = {}^A\tilde{\boldsymbol{\omega}}_{\boldsymbol{r}}{}^B \cdot \boldsymbol{T} + {}^A\tilde{\boldsymbol{v}}_{\boldsymbol{r}}{}^Q \cdot \boldsymbol{R} \quad (r = 1, \cdots, p) \tag{4.7}$$

4.7 Contributing Interaction Forces

There is contribution to $\tilde{F}_r$ if:

- two particles of a system are not rigidly connected to each other, the gravitational forces exerted by the particles on each other can make such contributions.
- bodies connected to each other by certain energy storage or energy dissipation devices.

4.8 Terrestrial Gravitational Forces

$$\boldsymbol{G_i} = m_i g \boldsymbol{k} \qquad (i = 1, \ldots, v) \tag{4.8}$$

$$(\tilde{F}_r)_\gamma = Mg\boldsymbol{k} \cdot \tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{*}} \tag{4.9}$$

$$(\tilde{F}_r)_\gamma = \sum_{i=1}^{v} \tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{P_i}} \cdot \boldsymbol{G_i} \tag{4.10}$$

where M is the total mass of S and $\tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{*}}$ is the rth partial velocity of the mass center of S in A.

4.9 Bridging Noncontributing Forces into Evidence

- Bring noncontributing force/torque of interest into evidence through the introduction of a generalized speed related to it.
- In effect, this permits points to have certain velocities or rigid bodies to have certain angular velocities which they cannot possess.
- Original generalized speeds and associated generalized active forces remain unaltered.

4.10 Coulomb Friction Forces

4.10.1 Particle P in contact with rigid body C

$$\boldsymbol{C} = N\boldsymbol{v} + T\boldsymbol{\tau} \tag{4.11}$$

$$|T| \leq \mu N \tag{4.12}$$

$$|T| = \mu N \qquad \text{impending tangential motion} \tag{4.13}$$

$$|T| = \mu' N \qquad \text{sliding} \tag{4.14}$$

$$\tag{4.15}$$

where N is nonnegative, $\boldsymbol{v}$ is the vector from C to P, $\boldsymbol{\tau}$ is perpendicular to $\boldsymbol{v}$, μ is the coefficient of static friction, μ' is the coefficient of kinetic friction.

4.10.2 Rigid body B in contact with rigid body C across area $\bar{A}$

$$d\boldsymbol{C} = (n\boldsymbol{v} + tvec\tau)dA \tag{4.16}$$

$$|t| \leq \mu n \tag{4.17}$$

$$|t| = \mu n \qquad \text{impending tangential motion} \tag{4.18}$$

$$|t| = \mu' n \qquad \text{sliding} \tag{4.19}$$

$$\tag{4.20}$$

where n is called the pressure at point P, t is called the shear at P.

4.11 Generalized Inertia Forces

$$\tilde{F}_r^* \triangleq \sum_{i=1}^{v} \tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{P_i}} \cdot \boldsymbol{R_i^*}, \ (r = 1, \ldots, p) \text{ nonholonomic} \tag{4.21}$$

$$F_r^* \triangleq \sum_{i=1}^{v} \boldsymbol{v}_{\boldsymbol{r}}^{\boldsymbol{P_i}} \cdot \boldsymbol{R_i^*}, \ (r = 1, \ldots, n) \text{ holonomic} \tag{4.22}$$

$$\boldsymbol{R_i^*} \triangleq -m_i \boldsymbol{a_i}, \ (i = 1, \ldots, v) \tag{4.23}$$

$$\tilde{F}_r^* = F_r^* + \sum_{s=p+1}^{n} F_s^* A_{sr} \tag{4.24}$$

$$\boldsymbol{T^*} \triangleq -\sum_{i=1}^{\beta} m_i \boldsymbol{r_i} \times \boldsymbol{a_i} \tag{4.25}$$

$$= -{}^A\boldsymbol{\alpha}^B \cdot \boldsymbol{I}^{B/B*} - {}^A\boldsymbol{\omega}^B \times \boldsymbol{I}^{B/B*} \cdot {}^A\boldsymbol{\omega}^B \tag{4.26}$$

$$\begin{aligned} &= -[\alpha_1 I_1 - \omega_2\omega_3(I_2 - I_3)]\boldsymbol{c_1} \\ &\quad - [\alpha_2 I_2 - \omega_3\omega_1(I_3 - I_1)]\boldsymbol{c_2} \\ &\quad - [\alpha_3 I_3 - \omega_1\omega_2(I_1 - I_2)]\boldsymbol{c_3} \end{aligned} \tag{4.27}$$

$$\boldsymbol{R^*} \triangleq -M\boldsymbol{a^*} \tag{4.28}$$

$$(\tilde{F}_r^*)_B = {}^A\tilde{\boldsymbol{\omega}}_{\boldsymbol{r}}{}^B \cdot \boldsymbol{T^*} + {}^A\tilde{\boldsymbol{v}}_{\boldsymbol{r}}^{\boldsymbol{*}B*} \cdot \boldsymbol{R^*}, \ (r = 1, \cdots, p) \tag{4.29}$$

5 Generalized Forces

5.1 Potential Energy

Nonholonomic (simple):

$$u_r \triangleq \dot{q}_r \ (r = 1, \ldots, n) \tag{5.1}$$

$$F_r = -\frac{\delta V}{\delta q_r} \tag{5.2}$$

$$0 = \frac{\delta V}{\delta t} \tag{5.3}$$

Nonholonomic (general):

$$u_r \triangleq \sum_{s=1}^{n} Y_{rs}\dot{q}_s + Z_r \tag{5.4}$$

$$\dot{q}_s = \sum_{r=1}^{n} W_{sr}u_r + X_s, \ (s = 1, \ldots, n) \tag{5.5}$$

$$F_r = -\sum_{s=1}^{n} \frac{\delta V}{\delta q_s} W_{sr} \tag{5.6}$$

$$0 = \frac{\delta V}{\delta t} + \sum_{s=1}^{n} \frac{\delta V}{\delta q_s} X_s \tag{5.7}$$

$$\dot{V} = -\sum_{r=1}^{n} F_r u_r \tag{5.8}$$

$$\frac{\delta}{\delta q_s}\frac{\delta V}{\delta q_r} = \frac{\delta}{\delta q_r}\frac{\delta V}{\delta q_s} \tag{5.9}$$

Holonomic (simple):

$$u_r \triangleq \dot{q_r} \; (r = 1, \ldots, n) \tag{5.10}$$

$$\dot{q_k} = \sum_{r=1}^{p} C_{kr}\dot{q_r} + D_k, \; (k = p+1, \ldots, n) \tag{5.11}$$

$$\tilde{F}_r = -(\frac{\delta V}{\delta q_r} + \sum_{s=p+1}^{n} \frac{\delta V}{\delta q_s} C_{sr}), \; (r = 1, \ldots, p) \tag{5.12}$$

$$0 = \frac{\delta V}{\delta t} + \sum_{k=p+1}^{n} \frac{\delta V}{\delta q_s} D_s \tag{5.13}$$

$$\tag{5.14}$$

Holonomic (general):

$$u_k \triangleq \sum_{r=1}^{p} A_{kr}u_r + B_k, \; (k = p+1, \ldots, n) \tag{5.15}$$

$$\dot{q_k} = \sum_{r=1}^{p} C_{kr}\dot{q_r} + D_k, \; (k = p+1, \ldots, n) \tag{5.16}$$

$$\tilde{F}_r = -\sum_{s=1}^{n} \frac{\delta V}{\delta q_s}(W_{sr} + \sum_{k=p+1}^{n} W_{sk}A_{kr}, \; (r = 1, \ldots, p) \tag{5.17}$$

$$0 = \frac{\delta V}{\delta t} + \sum_{s=1}^{n} \frac{\delta V}{\delta q_s}(X_s + \sum_{k=p+1}^{n} W_{sr}B_r) \tag{5.18}$$

$$\dot{V} = -\sum_{r=1}^{n} \tilde{F}_r u_r \tag{5.19}$$

Solving for V:

1. $f_{s-p} \triangleq \frac{\delta V}{\delta q_s}$ for $s = p+1, \ldots, n$.
2. Replace $\frac{\delta V}{\delta q_s}$ with f_{s-p}.
3. Form $\frac{\delta}{\delta q_j}\frac{\delta V}{\delta q_r}$
4. Rearrange to $ZX = Y$ where $X = \begin{bmatrix} \frac{\delta f_1}{\delta q_1} & \ldots & \frac{\delta f_1}{\delta q_n} & \ldots & \frac{\delta f_m}{\delta q_1} & \ldots & \frac{\delta f_m}{\delta q_n} \end{bmatrix}$
5. Get Reduced Row Echelon Form
6. Infer V
7. Substitute f_{s-p} into $\frac{\delta V}{\delta q_1} \ldots \frac{\delta V}{\delta q_s}$

5.2 Potential Energy Contributions

$$V \triangleq V_\alpha + V_\beta + \ldots \tag{5.20}$$

$$V_\gamma \triangleq -Mg\boldsymbol{k} \cdot \boldsymbol{p^*} \text{ gravity} \tag{5.21}$$

$$V_\sigma \triangleq \int_0^x f(\zeta)d\zeta \text{ spring (general)} \tag{5.22}$$

$$= \frac{1}{2}kx^2 \text{ linear spring} \tag{5.23}$$

5.3 Dissipation Functions

$\mathcal{F}$ is called a dissipation function for set C.

$$(\tilde{F}_r)_C = -\frac{\delta \mathcal{F}}{\delta u_r} \tag{5.24}$$

5.4 Kinetic Energy

- K_B = constribution of rigid body B to K of set S.
- K_ω = rotational kinetic energy of B in A.
- K_v = translational kinetic energy of B in A.

$$K \triangleq \frac{1}{2}\sum_{i=1}^{v} m_i(\boldsymbol{v}^{P_i})^2 \tag{5.25}$$

$$K_B = K_\omega + K_v \tag{5.26}$$

$$K_\omega = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega} \tag{5.27}$$

$$= \frac{1}{2}I\boldsymbol{\omega}^2 \tag{5.28}$$

$$= \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3} \omega_j I_{jk} \omega_k \tag{5.29}$$

$$= \frac{1}{2}\sum_{j=1}^{3} I_j \omega_j^2 \tag{5.30}$$

$$K_v = \frac{1}{2}m\boldsymbol{v}^2 \tag{5.31}$$

$$\tag{5.32}$$

5.5 Homogeneous Kinetic Energy Functions

$$K = K_0 + K_1 + K_2 \tag{5.33}$$

$$K_2 = \frac{1}{2}\sum_{r=1}^{p}\sum_{s=1}^{p} m_{rs}u_r u_s \tag{5.34}$$

$$m_{rs} \triangleq \sum_{i=1}^{v} m_i \tilde{\boldsymbol{v_r}}^{\boldsymbol{P_i}} \tilde{\boldsymbol{v_s}}^{\boldsymbol{P_i}}, \; (r, s = 1, \ldots, p) \tag{5.35}$$

$$= m_{sr} \tag{5.36}$$

5.6 Kinetic Energy and Generalized Inertia Forces

$$\dot{K_2} - \dot{K_0} = -\sum_{r=1}^{p} \tilde{F}_r^* u_r \tag{5.37}$$

is satisfied iff.

$$\sum_{i=1}^{v} m_i \boldsymbol{v^{P_i}} \cdot \dot{\tilde{\boldsymbol{v}}}^{P_i} = 0 \tag{5.38}$$

OR

$$\frac{\delta K}{\delta t} + \sum_{s=1}^{n} \frac{\delta K}{\delta q_s}\left(X_s + \sum_{r=p+1}^{n} W_{sr}B_r\right) = 0 \tag{5.39}$$

$$\frac{\delta}{\delta t}\left(X_s + \sum_{r=p+1}^{n} W_{sr}B_r\right) = 0, \; (s = 1, \ldots, n) \tag{5.40}$$

$$\frac{\delta X_s}{\delta t} + \sum_{k=1}^{n} \frac{\delta W_{sk}}{\delta t}u_k = \frac{\delta X_s}{\delta qr} + \sum_{k=1}^{n} \frac{\delta W_{sk}}{\delta q_r}u_k = 0, \; (r, s = 1, \ldots, n) \tag{5.41}$$

If K is function of $q_1, \ldots, q_n$ and $\dot{q}_1, \ldots, \dot{q}_n$, then

$$\tilde{F}_r^* = -\sum_{s=1}^{n}\left(\frac{d}{dt}\frac{\delta K}{\delta \dot{q}_s} - \frac{\delta K}{\delta q_s}\right)\left(W_{sr} + \sum_{k=p+1}^{n} W_{sk}A_{kr}\right) \tag{5.42}$$

$$F_r^* = -\sum_{s=1}^{n}\left(\frac{d}{dt}\frac{\delta K}{\delta \dot{q}_s} - \frac{\delta K}{\delta q_s}\right)W_{sr} \tag{5.43}$$

If $u_r = \dot{q}_r$,

$$\tilde{F}_r^* = -\left[\frac{d}{dt}\frac{\delta K}{\delta \dot{q}_r} - \frac{\delta K}{\delta q_r} + \sum_{s=p+1}^{n}\left(\frac{d}{dt}\frac{\delta K}{\delta \dot{q}_s} - \frac{\delta K}{\delta q_s}\right) C_{sr}\right] \tag{5.44}$$

$$F_r^* = \left(\frac{d}{dt}\frac{\delta K}{\delta \dot{q}_r} - \frac{\delta K}{\delta q_r}\right) \tag{5.45}$$