MIT 16.322 Stochastic Estimation and Control

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October 21, 2019

Studying based on the book *Introduction to Random Signals and Applied Kalman Filtering WITH MATLAB EXERCISES 4th Edition by Robert Grover Brown and Patrick Hwang*

1 Random Signals Background

1.1 Lecture 01 - 08 (1.1 - 1.17)

1.1 In straight poker, five cards are dealt to each player from a deck of ordinary playing cards. What is the probability that a player will be dealt a flush (i.e., five cards all of one suit)?

$\frac{4*13*12*11*10*9}{52*51*50*49*48} = \frac{33}{16660}$

1.2 In the game of blackjack, the player is initially dealt two cards from a deck of ordinary playing cards. Without going into all the details of the game, it will suffice to say here that the best possible hand one could receive on the initial deal is a combination of an ace of any suit and any face card or 10. What is the probability that the player will be dealt this combination?

$\frac{4*16+16*4}{52*51} = \frac{32}{663}$

1.3a (refer to the table in the book) If not joint probabilities, what do the numerical values represent in terms of probabilities (if anything)?

I believe the numerical values are conditional probabilities, Pr(sentiment | political belief). Eg, Pr(sentiment = Strongly in favor | political belief = Democratic) = 0.36.

1.3b If we have two sets of random outcomes, not necessarily disjoint, and we have the table of joint probabilities, then we can always get all of the conditional and unconditional probabilities from the joint table. This is to say that the joint table tells the "whole story." Can the table of joint probabilities be obtained from the table given in this problem? Explain your answer.

We can use the relationship between joint probabilities to infer about 1 set of the random outcomes, but only for 1 (not 2) since there is no information between the 2nd set of outcomes in the table.

1.4 Imagine a simple dice game where three dice are rolled simultaneously. Just as in craps, in this game we are only interested in the sum of the dots on any given roll.

1.4a Describe, in words, the sample space for this probabilistic scenario. (You do not need to calculate all the associated probabilities—just a few at the low end of the realizable sums will suffice.)

It can be line where you have points from integers 3 to 18.

1.4b is the probability of rolling a 3?

$\frac{1}{6} * \frac{1}{6} * \frac{1}{6} = \frac{1}{216}$

1.4c What is the probability of rolling a 4?

$\frac{1+1+1}{6*6*6} = \frac{3}{216}$ (possible dice roll: 1,1,2 or 1,2,1 or 2,1,1)

1.4d What is the probability of rolling a 3 or 4?

$\frac{1+3}{216} = \frac{1}{54}$ (sum of the two answers above. note that they are mutually exclusive)

1.8 Cribbage is an excellent two-player card game. It is played with a 52-card deck of ordinary player cards. The initial dealer is determined by a cut of the cards, and the dealer has a slight advantage over the nondealer because of the order in which points are counted. Six cards are dealt to each player and, after reviewing their cards, they each contribute two cards to the crib. The non-dealer then cuts the remaining deck for the dealer who turns a card. This card is called the starter, if it turns out to be a jack, the dealer immediately scores 2 points. Thus, the jack is an important card in cribbage.

1.8a What is the unconditional probability that the starter card will be a jack?

$\frac{4}{52} = \frac{1}{13}$

1.8bNow consider a special case where both the dealer and non-dealer have looked at their respective six cards, and the dealer observes that there is no jack in his hand. At this point the deck has not yet been cut for the starter card. From the dealer's viewpoint, what is the probability that the starter card will be a jack?

$\frac{4}{46} = \frac{2}{23}$

1.8c Now consider another special case where the dealer, in reviewing his initial six cards, notes that one of them is a jack. Again, from his vantage point, what is the probability that the starter card will be a jack?.

$\frac{4}{46} = \frac{2}{23}$. Assuming the dealer plays optimally, he can put the Jack back into the deck entering the same situation in 1.8b.

1.9 Assume equal likelihood for the birth of boys and girls. What is the probability that a four-child family chosen at random will have two boys and two girls, irrespective of the order of birth? Note: The answer is not 1/2 as might be suspected at first glance.

$\frac{\binom{4}{2}}{2*2*2*2} = \frac{2*3}{2*2*2*2} = \frac{3}{8}$

1.10 Consider a sequence of random binary digits, zeros and ones. Each digit may be thought of as an independent sample from a sample space containing two elements, each having a probability of $\frac{1}{2}$. For a six-digit sequence, what is the probability of having:

1.10a Exactly 3 zeros and 3 ones arranged in any order?

$\frac{\binom{6}{3}}{2^6} = \frac{5}{16}$

1.10b Exactly 4 zeros and 2 ones arranged in any order?

$\frac{\binom{6}{4}}{2^6} = \frac{15}{64}$

1.10c Exact 5 zeros and 1 one arranged in any order?

$\frac{\binom{6}{5}}{2^6} = \frac{3}{32}$

1.10d Exactly 6 zeros?

$\frac{1}{2^6} = \frac{1}{64}$

1.12 Video poker has become a popular game in casinos in the United States (4). The player plays against the machine in much the same way as with slot machines, except that the machine displays cards on a video screen instead of the familiar bars, etc., on spinning wheels. When a coin is put into the machine, it immediately displays five cards on the screen. After this initial five-card deal, the player is allowed to discard one to five cards at his or her discretion and obtain replacement cards (i.e., this is the "draw"). The object, of course, is to try to improve the poker hand with the draw.

1.12a Suppose the player is dealt the 3, 7, 8, 10 of hearts and the queen of spades on the initial deal. The player then decides to keep the four hearts and discard the queen of spades in hopes of getting another heart on the draw, and thus obtain a flush (five cards, all of the same suit). The typical video poker machine pays out five coins for a flush. Assume that this is the payout and that the machine is statistically fair. What is the expected (i.e., average) return for this draw situation? (Note that an average return of 1.0 is the break-even return.)

$5 * Pr(Win) = 5 * \frac{9}{48} = \frac{15}{16}$

1.12b Some of the Las Vegas casinos advertise 100 percent (or "full pay") video poker machines. These machines have pay tables that are such that the machines will return slightly greater than 100 percent under the right conditions. How can this be? The answer is that most of the players do not play a perfect game in terms of their choices on the draw. Suppose, hypothetically, that only 10 percent of the players make perfect choices on the draw and they achieve a 100.2 percent; and the other 90 percent of the players only achieve a 98 percent return. What would be the casino percentage return under these circumstances?

$0.1 * 1.002 + 0.9 * 0.98 = 0.9822$

1.13 The random variable X may take on all values between 0 and 2, with all values within this range being equally likely. Calculate $E(X)$, $E(X^2)$, and $VarX$

$E(X) = \int_0^2 x * \frac{1}{2} dx = \frac{1}{2} * \frac{x^2}{2}\Big|_0^2 = 1$ $E(X^2) = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2} * \frac{x^3}{3}\Big|_0^2 = \frac{4}{3}$ $VarX = E(X^2) - E(X)^2 = \frac{4}{3} - 1^2 = \frac{1}{3}$

1.14 A random variable X has a probability density function as shown. What is the variance of X?

$E(X) = \int_0^2 x * \frac{x}{2} dx = \frac{x^3}{2*3}\Big|_0^2 = \frac{4}{3}$ $E(X^2) = \int_0^2 x^2 * \frac{x}{2} dx = \frac{x^4}{2*4}\Big|_0^2 = 2$ $VarX = E(X^2) - E(X)^2 = 2 - (\frac{4}{3})^2 = \frac{2}{9}$

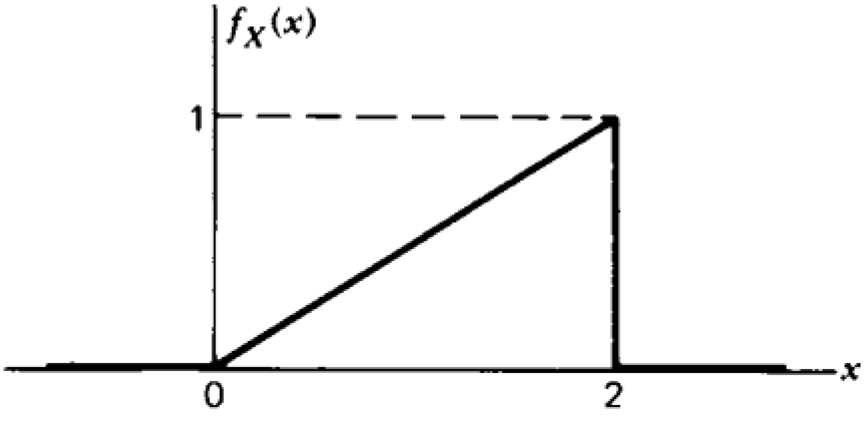

Figure 1: Figure p1.14

1.15 A random variable X whose probability density function is given by $f_X(x) = \alpha e^{-\alpha x}$ for $x \geq 0$ and 0 otherwise is said to have an exponential probability density function. This density function is sometimes used to describe the failure of equipment components (13). That is, the probability that a particular component will fail within time T is $P(failure) = \int_0^T f_X(x)dx$. Note that α is a parameter that may be adjusted to fit the situation at hand. Find α for an electronic component whose average lifetime is 10,000 hours. ("Average" is used synonymously with "expectation" here.)

Note that we will use integration by parts for $\int xe^x dx = xe^x - \int e^x dx$.

$$
\begin{aligned}
10000 &= \int_0^\infty x f_X(x)dx = \int_0^\infty \alpha x e^{-\alpha x} dx \\
10000 &= -xe^{-\alpha x}\Big|_0^\infty - \frac{1}{\alpha}\int_0^\infty -\alpha e^{-\alpha x} dx \\
10000 &= 0 - \frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha} \\
\alpha &= \frac{1}{10000}
\end{aligned}
$$

1.16 Consider a sack containing several identical coins whose sides are labeled $+1$ and -1. A certain number of coins are withdrawn and tossed simultaneously. The algebraic sum of the numbers resulting from the toss is a discrete random variable. Sketch the probability density function associated with the random variable for the following situations: 1 coin, 2 coins, 5 coins, 10 coins

In general, we have $P(sum = i) = \frac{\binom{n}{i}}{2^n}$ where n is the number of coins.

1.18 Discrete random variables X and Y may each take on integer values 1, 3, and 5, and the joint probability of X and Y is given in the table below.

1.18a Are random variables X and Y independent?

No. $P(X = 1 and Y = 1) = \frac{1}{18}$, $P(X = 1) = P(Y = 1) = \frac{3}{18}$ but $P(X = 1 and Y = 1) \neq P(X = 1) * P(Y = 1)$.

X \ Y	1	3	5
1	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
2	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{6}$
3	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{3}$

Figure 2: Figure p1.18

1.18b Find the unconditional probability P(Y=5)

$P(Y=5) = \frac{1}{18} + \frac{1}{6} + \frac{1}{3} = \frac{5}{9}$

1.18c What is the conditional probability P(Y=5 | X = 3)?

$P(Y=5|X=3) = \frac{P(Y=5andX=3)}{P(X=3)} = \frac{\frac{1}{6}}{\frac{1}{18}+\frac{1}{18}+\frac{1}{6}} = \frac{3}{5}$

1.19 The diagram shown as Fig. P1.19 gives the error characteristics of a hypothetical binary transmission system. The numbers shown next to the arrows are the conditional probabilities of Y given X. The unconditional probabilities for X are shown to the left of the figure. Find:

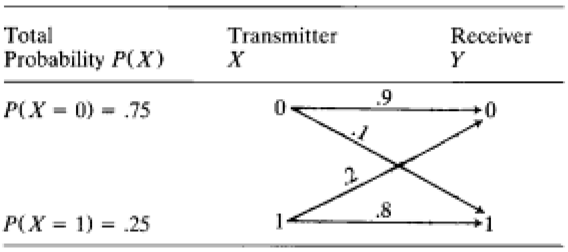

Figure P1.19

1.19a The conditional probabilities $P(X=0|Y=1)$ and $P(X=0|Y=0)$.

$P(X=0|Y=1) = \frac{P(Y=1|X=0)*P(X=0)}{P(Y=1)} = \frac{0.1*0.75}{0.275} = \frac{3}{11}$ $P(X=0|Y=0) = \frac{P(Y=0|X=0)*P(X=0)}{P(Y=0)} = \frac{0.9*0.75}{0.725} = \frac{27}{29}$

1.19b The unconditional probabilities $P(Y=0)$ and $P(Y=1)$

$P(Y=0) = P(Y=0|X=0)*P(X=0) + P(Y=0|X=1)*P(X=1) = 0.9*0.75 + 0.2*0.25 = 0.725$ $P(Y=1) = P(Y=1|X=0)*P(X=0) + P(Y=1|X=1)*P(X=1) = 0.1*0.75 + 0.8*0.25 = 0.275$

1.19c The joint probability array for $P(X,Y)$

$P(X,Y) = P(Y|X)*P(X)$

1.20 The Rayleigh probability density function is defined as $f_R(r) = \frac{r}{\sigma^2}e^{-r^2/2\sigma^2}$ where σ^2 is a parameter of the distribution.

1.20a Find the mean and variance of a Rayleigh distributed random variable R

1. Solving for the mean, we did integration by parts ($\int udv = uv - \int vdu$) where $u = x$, $du = dx$, $v = -e^{-x^2/2\sigma^2}$, and $dv = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2}$. Note that the Rayleigh distribution is only valid for $x \geq 0$.

$$
\begin{aligned}
E[X] &= \int_0^\infty \frac{x^2}{\sigma^2}e^{-x^2/2\sigma^2}dx \\
&= -xe^{-x^2/2\sigma^2}\Big|_0^\infty - \frac{1}{2}\int_{-\infty}^{\infty} -e^{-x^2/2\sigma^2}dx \\
&= 0 + \frac{\sqrt{2\pi}\sigma}{2} = \sqrt{\frac{\pi}{2}}\sigma
\end{aligned}
$$

2. Solving for the variance, again, we used integration by parts ($\int udv = uv - \int vdu$) where $u = x^2$, $du = 2xdx$, $v = -e^{-x^2/2\sigma^2}$, and $dv = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2}dx$.

$$
\begin{aligned}
E[x^2] &= \int_0^\infty \frac{x^3}{\sigma^2}e^{-x^2/2\sigma^2}dx \\
&= -x^2e^{-x^2/2\sigma^2}\Big|_0^\infty - \frac{2\sigma^2}{\sigma^2}\int_0^\infty -xe^{-x^2/2\sigma^2}dx \\
&= 0 - 2\sigma^2e^{-x^2/2\sigma^2}\Big|_0^\infty = 2\sigma^2 \\
VarX &= E[x^2] - E[x]^2 \\
&= 2\sigma^2 - \frac{\pi}{2}\sigma^2 = \frac{4-\pi}{2}\sigma^2
\end{aligned}
$$

1.20b Find the mode of R (i.e., the most likely value of R).

Mode of R is the value of r such that $f_R(r)$ is highest. We can solve this by taking the derivative of $f_R(r)$ and equating it to zero.

$$
\begin{aligned}
\frac{df_R(r)}{dr} = 0 &= \frac{1}{\sigma^2}e^{-r^2/2\sigma^2} - \frac{r^2}{\sigma^4}e^{-r^2/2\sigma^2} \\
&= 1 - \frac{r^2}{\sigma^2} \\
r &= \pm\sigma
\end{aligned}
$$

By trying the two possible values of r, we can see that maximum will occur on $r = \sigma$ hence that will be our mode.

1.21 The target shooting example of Section 1.13 led to the Rayleigh density function specified by Eq. (1.13.22) $f_R(r) = \frac{r}{\sigma^2}e^{-r^2/2\sigma^2}$

1.21a Show that the probability that a hit will lie within a specified distance R_o from the origin is given by $P(HitlieswithinR_o) = 1 - e^{-R_o^2/2\sigma^2}$

$$
\begin{aligned}
P(HitlieswithinR_o) &= \int_0^{R_o} \frac{r}{\sigma^2}e^{-r^2/2\sigma^2}dr \\
&= -e^{-r^2/2\sigma^2}\Big|_0^{R_o} \\
&= 1 - e^{-R_o^2/\sigma^2}
\end{aligned}
$$

1.21b The value of R0 in Eq. (P1.21) that yields a probability of .5 is known as the circular probable error (or circular error probable, CEP). Find the CEP in terms of σ.

$$\begin{aligned} P(Hitlieswithin R_o) &= 1 - e^{-R_o^2/\sigma^2} \\ 0.5 &= 1 - e^{-R_o^2/\sigma^2} \\ e^{-R_o^2/\sigma^2} &= 0.5 \\ -R_o^2/\sigma^2 &= \ln(0.5) \\ R_o &= \pm\sqrt{\ln 2}\sigma \end{aligned}$$

1.21c Navigation engineers also frequently use a 95 percent accuracy figure in horizontal positioning applications. (This is in contrast to CEP.) The same circular symmetry assumptions used in parts (a) and (b) apply here in computing the 95 percent radius. To be specific, find the R_{95} which is such that 95 percent of the horizontal error lies within a circle of radius R_{95}. Just as in part (b) express R_{95} in terms of s.

$$\begin{aligned} P(Hitlieswithin R_o) &= 1 - e^{-R_o^2/\sigma^2} \\ 0.95 &= 1 - e^{-R_o^2/\sigma^2} \\ e^{-R_o^2/\sigma^2} &= 0.05 \\ -R_o^2/\sigma^2 &= \ln(0.05) \\ R_o &= \pm\sqrt{\ln 20}\sigma \end{aligned}$$

1.22 Consider a random variable X with an exponential probability density function $f_X(x) = e^{-x}$ if $x \geq 0$, and 0 if $x < 0$.

1.22a $P(X \geq 2)$

$\int_2^\infty e^{-x} dx = -e^{-x}\Big|_2^\infty = e^{-2}$

1.22b $P(1 \leq X \leq 2)$

$\int_1^2 e^{-x} dx = -e^{-x}\Big|_1^2 = e^{-1} - e^{-2}$

1.22c $E(X)$**,** $E(X^2)$**, and** $VarX$**.**

$E(X) = \int_0^\infty xe^{-x} dx = -xe^{-x}\Big|_0^\infty - \int_0^\infty -e^{-x} dx = 0 - e^{-x}\Big|_0^\infty = 1$ $E(X^2) = \int_0^\infty x^2 e^{-x} dx = -x^2 e^{-x}\Big|_0^\infty + 2\int_0^\infty xe^{-x} dx = 0 + 2 = 2$ $VarX = E(X^2) - E(X)^2 = 2 - 1^2 = 1$

1.23 Random variables X and Y have a joint probability density function defined as follows: $f_{XY}(x, y) = 0.25$ if $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, and 0 otherwise. Are random variables X and Y statistically independent?

$f_X(x) = \int_{-1}^1 0.25 dy = 0.5$ $f_Y(y) = \int_{-1}^1 0.25 dx = 0.5$ Because $f_{XY} = f_X * f_Y$, we can conclude that X and Y are statistically independent.

1.24 Random variables X and Y have a joint probability density function $f_{XY}(x, y) = e^{-(x+y)}$ if $x \geq 0$ and $y \geq 0$, and 0 otherwise. Find $P(X \leq 1/2)$, $P((X+Y) \leq 1)$, $P[(XorY) \geq 1]$, and $P[(XandY) \geq 1]$.

$$\begin{aligned}
P(X \leq \frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_0^{\infty} e^{-(x+y)} dy dx \\
&= \int_0^{\frac{1}{2}} -e^{-(x+y)} \Big|_0^{\infty} dx \\
&= \int_0^{\frac{1}{2}} e^{-x} dx \\
&= -e^{-x} \Big|_0^{0.5} = 1 - e^{-0.5} = 0.3935
\end{aligned}$$

$$\begin{aligned}
P((X+Y) \leq 1) &= \int_0^1 \int_0^{1-x} e^{-(x+y)} dy dx \\
&= \int_0^1 -e^{-(x+y)} \Big|_0^{1-x} dx \\
&= \int_0^1 1 - e^{-1} dx = 1 - e^{-1} = 0.6321
\end{aligned}$$

$$\begin{aligned}
P[(XorY) \geq 1] &= \int_0^1 \int_1^{\infty} e^{-(x+y)} dy dx + \int_1^{\infty} \int_0^1 e^{-(x+y)} dy dx + \int_1^{\infty} \int_1^{\infty} e^{-(x+y)} dy dx \\
&= \int_0^1 \int_1^{\infty} e^{-(x+y)} dy dx + \int_1^{\infty} \int_0^{\infty} e^{-(x+y)} dy dx \\
&= \int_0^1 -e^{-(x+y)} \Big|_1^{\infty} dx + \int_1^{\infty} -e^{-(x+y)} \Big|_0^{\infty} dx \\
&= \int_0^1 e^{-(x+1)} dx + \int_1^{\infty} e^{-x} dx \\
&= -e^{-(x+1)} \Big|_0^1 + -e^{-x} \Big|_1^{\infty} \\
&= e^{-1} - e^{-2} + e^{-1} = 0.6
\end{aligned}$$

$$\begin{aligned}
P[(XandY) \geq 1] &= \int_1^{\infty} \int_1^{\infty} e^{-(x+y)} dy dx \\
&= \int_1^{\infty} -e^{-(x+y)} \Big|_1^{\infty} dx \\
&= \int_1^{\infty} e^{-(x+1)} dx \\
&= -e^{-(x+1)} \Big|_1^{\infty} = -0 + e^{-2} = 0.135
\end{aligned}$$

1.26 Random variables X and Y are statistically independent and their respective probability density functions are $f_X(x) = \frac{1}{2}e^{-|x|}$ and $f_Y(y) = e^{-2|y|}$. Find the probability density function associated with $X + Y$.

Note $\mathscr{F}[e^{-a|t|}] = \frac{2a}{\omega^2+a^2}$ (from Table A.2).

$$\begin{aligned}
\mathscr{F}_X &= \frac{1}{2} * \frac{2}{\omega^2+1} \\
\mathscr{F}_Y &= \frac{2*2}{\omega^2+4} \\
\mathscr{F}_Z &= \mathscr{F}_X * \mathscr{F}_Y \\
&= \frac{4}{(\omega^2+1)(\omega^2+4)} \\
&= \frac{4/3}{\omega^2+1} - \frac{4/3}{\omega^2+2^2} \\
f_Z(z) &= \frac{2}{3}e^{-|t|} - \frac{1}{3}e^{-2|t|}
\end{aligned}$$

1.27 Random variable X has a probability density function $f_X(x) = \frac{1}{2}$ if $-1 \le x \le 1$ and 0 otherwise. Random variable Y is related to X through the equation $y = x^3 + 1$. What is the probability density function of Y?

Note: $f_Y(y) = |h'(y)|f_X(h(y))$, $h(y) = (y-1)^{1/3}$, $h'(y) = \frac{1}{3}(y-1)^{-2/3}$ $f_Y(y) = \frac{1}{6}|(y-1)^{-2/3}|$ given $0 \le y \le 2$

1.28 Find the mean and covariance matrix Y given X, m_X, C_X, and $y = Ax + b$.

Skipping this problem. Key equations below. $m_Y = Am_X + b$ $C_Y = A * C_X * A^T$

1.29 A pair of random variables, X and Y, have a joint probability density function $f_{XY}(x, y) = 1$ if $0 \le y \le 2x$ and $0 \le x \le 1$, 0 otherwise. Find $E(X|Y = 0.5)$.

$$\begin{aligned}
f_{XY}(x, y = 0.5) &= 1, if 0.25 \le x \le 1 \\
f_Y(y) &= \int_{y/2}^{1} 1dx = (1 - \frac{y}{2}) \\
f_Y(y = 0.5) &= (1 - \frac{0.5}{2}) = 0.75 \\
f_{X|Y=0.5}(x) &= \frac{f_{XY}(x, y = 0.5)}{f_Y(y = 0.5)} \\
&= \frac{4}{3}, if 0.25 \le x \le 1 \\
E(X|Y = 0.5) &= \int_{0.25}^{1} x * \frac{4}{3}dx = \frac{4}{3}\frac{x^2}{2}\Big|_{0.25}^{1} \\
&= \frac{4}{3} * \frac{1-0.25^2}{2} = \frac{5}{8}
\end{aligned}$$

1.30 Two continuous random variables X and Y have a joint probability density function that is uniform inside the unit circle and zero outside, that is,

$$f_{XY}(x, y) = \begin{cases} 1/\pi, & (x^2 + y^2) \le 1 \\ 0, & (x^2 + y^2) > 1 \end{cases}$$

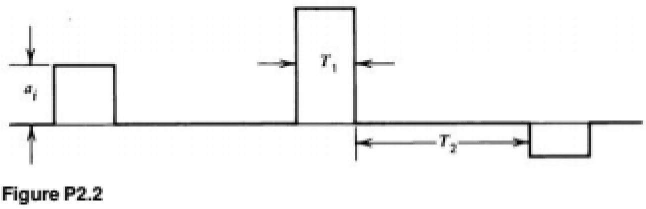

Figure 3: P2.2

1.30a Find the unconditional probability density function for the random variable Y and sketch the probability density as a function of Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$
$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx$$
$$= \frac{2\sqrt{1-y^2}}{\pi}, ify^2 \leq 1, 0otherwise$$

1.30b Are the random variables X and Y statistically independent?

No. From above we can solve $f_X(x) = \frac{2\sqrt{1-x^2}}{\pi}, ifx^2 \leq 1, 0otherwise$. Independence means $f_{XY} = f_X f_Y$ but our equations doesn't give us this.

2 Mathematical Description of Random Signals

2.1 Lecture 09 - 13 (2.1 - 2.14)

2.1 Noise measurements at the output of a certain amplifier (with its input shorted) indicate that the rms output voltage due to internal noise is $100\mu V$. If we assume that the frequency spectrum of the noise is flat from 0 to 10 MHz and zero above 10 MHz, find the a) spectral density function for the noise and b) autocorrelation function for the noise.

From section 2.8 white noise, we know that $S(jw) = A$ for $|\omega| \leq 2\pi 10 * 10^6 Hz$ and 0 otherwise, and $R(\tau) = 2 * 10 * 10^6 * A\frac{sin(2\pi W\tau)}{2\pi W\tau}$. Given that rms or $\sqrt{E[X^2]}$ or $\sqrt{R(0)}$ is $100\mu V$, we can solve for A.

$$\sqrt{R(0)} = \sqrt{2 * 10 * 10^6 * A\frac{sin(2\pi W * 0)}{2\pi W * 0}}$$
$$100 * 10^-6 = \sqrt{2 * 10 * 10^6 * A * 1}$$
$$A = 5 * 10^-16V$$

Note: I am not entirely sure whether I am using the correct units, but I am confident that my solution does follow the correct flow. We can plug in A back to $S(j\omega)$ and $R(\tau)$ to obtain the spectral density function and autocorrelation function.

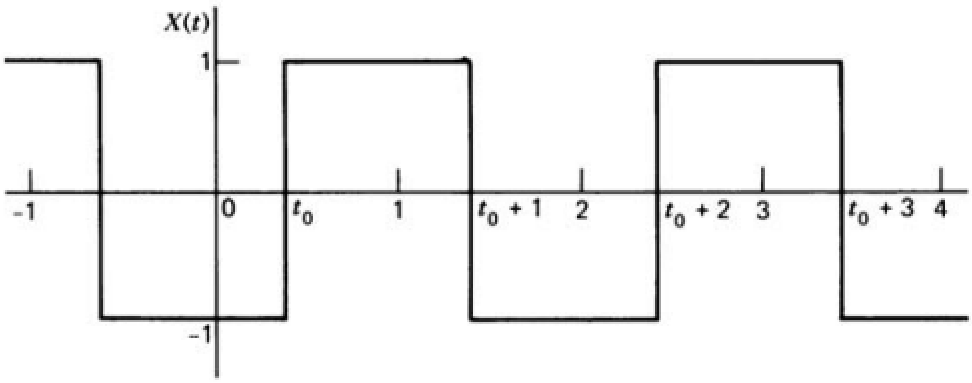

Figure P2.8

2.2 A sketch of a sample realization of a stationary random process X(t) is shown in the figure. The pulse amplitudes a_i are independent samples of a normal random variable with zero mean and variance σ^2. The time origin is completely random. Find the autocorrelation function for the process.

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{2\pi\sigma^2} e^{\frac{-(t^2+(t+\tau)^2)}{2\sigma^2}} dt$$
$$= \frac{1}{2\pi\sigma} e^{-\frac{\tau^2}{4\sigma^2}} \lim_{T\to\infty}\int_0^T \frac{1}{\sigma} e^{-\frac{(t+\tau/2)^2}{\sigma^2}} dx$$
$$= \frac{1}{2\pi\sigma} e^{-\frac{\tau^2}{4\sigma^2}} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4\pi\sigma} e^{-\frac{\tau^2}{4\sigma^2}}$$

2.3 Find the autocorrelation function corresponding to the spectral density function $S(j\omega) = \delta(\omega) + \frac{1}{2}\delta(\omega-\omega_0) + \frac{1}{2}\delta(\omega+\omega_0) + \frac{2}{\omega^2+1}$

Using Fourier tables, we can obtain: $\mathscr{F}[S(j\omega)] = \frac{1}{2\pi} + \frac{1}{2\pi} cos(\omega_0 t) + e^{-|t|}$

2.4 A stationary Gaussian random process X(t) has an autocorrelation function of the form $R_X(\tau) = 4e^{-|\tau|}$. What fraction of the time will $|X(t)|$ exceed four units?

Observing $R_X(\infty) = 0$, we can say that $X(t)$ is zero mean ($\mu = 0$). Note that $E[X^2] = \sigma^2 = R_X(0) = 4$. Given that $X(t)$ is a Gaussian random process with $\mu = 0$ and $\sigma^2 = 4$, using Casio fx 991 calculator's normal distribution feature $Pr(|X(t)| > 4) = P(\frac{-4-0}{2}) + R(\frac{4-0}{2}) = 0.0455$. Note: $P(x)$ is probability sum from $-\infty \to x$, while $R(x)$ is probability sum from $x \to \infty$.

2.8 A sample realization of a random process $X(t)$ is shown in the figure. The time t_0 when the transition from the -1 state to the $+1$ state takes place is a random variable that is uniformly distributed between 0 and 2 units.

(a) Is the process stationary? Yes. $Y = X(t_1)$ does not depend on t_1. $f_Y(a) = 0.5$ for $a = -1, 1$ and 0 otherwise. (b) Is the process deterministic or nondeterministic? The process is deterministic. Once t_0 is set, the rest falls into place. (c) Find the autocorrelation function and spectral density function for the process. Note that $R_X(\tau) = E[X(t_1)X(t_1+\tau)]$. If $\lceil\tau\rceil \mod 2 = 1$, then the amplitude of t_1 will always be the opposite of $t_1+\tau$ ($R_X(\tau) = 1 * -1 * 0.5 + -1 * 1 * 0.5$). If $\lceil\tau\rceil \mod 2 = 0$, then the amplitude of t_1 will be the same as $t_1+\tau$ ($R_X(\tau) = 1*1*0.5 + -1*-1*0.5$).

2.9 A common autocorrelation function encountered in physical problems is $R(\tau) = \sigma^2 e^{-\beta|\tau|} cos\omega_o\tau$. Find the corresponding spectral density function.

From the Fourier table: $e^{-a|t|}\cos bt = \frac{a}{a^2+(\omega+b)^2} + \frac{a}{a^2+(\omega-b)^2}$. $\mathscr{F}[R(\tau)] = \sigma^2 * (\frac{\beta}{\beta^2+(\omega+\omega_0)^2} + \frac{\beta}{\beta^2+(\omega-\omega_0)^2})$

2.10 Show that a Gauss–Markov process described by the autocorrelation function $R(\tau) = \sigma^2 e^{-\beta|\tau|}$ becomes Gaussian white noise if we let $\beta \to \infty$ and $\sigma^2 \to \infty$ in such a way that the area under the autocorrelation-function curve remains constant in the limiting process

By definition, a Gauss-Markov process is Gaussian. Therefore, we only need to show that the said process becomes white noise given the said transformations. From the Fourier table: $e^{-a|t|} = \frac{2a}{\omega^2+a^2}$.

$$\begin{aligned}\mathscr{F}[R(\tau)] &= \sigma^2 * \frac{2\beta}{\omega^2+\beta^2} \\ &= \lim_{x\to\infty} \frac{x*2*x}{\omega^2+x^2} = \lim_{x\to\infty} \frac{2x}{2x} = 1\end{aligned}$$

Note that since β and σ^2 approaches ∞, I substituted them to x, then applying Lhopitals rule, we get a constant power spectral density, which proves that the Gauss Markov process becomes Gaussian white noise.

2.11 A stationary random process X(t) has a spectral density function of the form $S_X(\omega) = \frac{6\omega^2+12}{(\omega^2+4)(\omega^2+1)}$.What is the mean-square value of X(t)?

Note from Fourier table: $e^{-a|t|} = \frac{2a}{\omega^2+a^2}$

$$\begin{aligned}S_X(\omega) &= \frac{4}{\omega^2+4} + \frac{2}{\omega^2+1} \\ \mathscr{F}^{-}[S_X(\omega)] &= R_X(\tau) = e^{-2|t|} + e^{-|t|} \\ E[X^2] &= R_X(0) = 2\end{aligned}$$

2.12 The stationary process $X(t)$ has an autocorrelation function of the form $R(\tau) = \sigma^2 e^{-\beta|\tau|}$. Another process $Y(t)$ is related to $X(t)$ by the deterministic equation $Y(t) = aX(t) + b$ where a and b are known constants. What is the autocorrelation function for $Y(t)$?

Note that $E[X(t)X(t+\tau)] = R(\tau)$ and that $R(\tau)$ describes a Gaussian Markov process, meaning that $E[X(t)] = 0$.

$$\begin{aligned}E[Y(t)Y(t+\tau)] &= E[(aX(t)+b)(aX(t+\tau)+b)] \\ &= E[a^2X(t)X(t+\tau) + abX(t) + abX(t+\tau) + b^2] \\ &= a^2R(\tau) + b^2\end{aligned}$$

2.13 The discrete random walk process is discussed in Section 2.11. Assume each step is of length l and that the steps are independent and equally likely to be positive or negative. Show that the variance of the total distance D traveled in N steps is given by $VarD = l^2N$.

Note that $D = l_1 + ... + l_N$, where $l_1, ..., l_N$ are all independent. Also, $E[l_i] = +l*0.5 + -l*0.5 = 0$, $E[(l_i)^2] = (+l)^2*0.5 + (-l)^2*0.5 = l^2$, and $E[l_il_j] = 0$ for $i \neq j$.

$$\begin{aligned}E[D] &= E[l_1 + ... + l_N] \\ &= E[l_1] + ... + E[l_N] = 0 \\ VarD &= E[(D - E[D])^2] = E[(l_1 + ... + l_N)^2] \\ &= E[(l_1)^2] + ... + E[(l_N)^2] = l^2N\end{aligned}$$

2.14 The Wiener process was discussed in Section 2.11. It is defined to be Gaussian random walk that begins with zero at $t = 0$. A more general random walk process can be defined to be similar to the Wiener process except that it starts with a $N(0, \sigma^2)$ random variable where σ is specified.

(a) What is the autocorrelation function for this more general process? Denote the white noise PSD driving the integrator as W.

$$\begin{aligned}
R_X(t_1, t_2) = E[X(t_1)X(t_2)] &= E[\int_{T0_1}^{t_1} W(u)du \int_{T0_2}^{t_2} W(v)dv] \\
&= \int_{T0_1}^{t_1}\int_{T0_2}^{t_2} E[W(u)W(v)]dudv \\
&= t_2 - T0_2 if t_1 - T0_1 \geq t_2 - T0_2 else t_1 - T0_1 if t_1 - T0_1 < t_2 - T0_2 \\
&= t_2 if t_1 \geq t_2 else t_1 if t_1 < t_2
\end{aligned}$$

Note that the mean of $T0_1$ and $T0_2$ is 0.

(b) Write the expression for the mean-square value of the process as a function of W, σ, and the elapsed time from $t = 0$.

$$\begin{aligned}
E[X(t)^2] &= E[\int_{T}^{t} W(u)du \int_{T}^{t} W(v)dv] \\
&= \int_{T}^{t}\int_{T}^{t} E[W(u)W(v)]dudv \\
&= t - T = t
\end{aligned}$$

Note that the mean of T is 0.

2.16 The spectral density function for the stationary process $X(t)$ is $S_X(j\omega) = \frac{1}{(1+\omega^2)^2}$. Find the autocorrelation function for $X(t)$.

Note: $(f * g)(t) = F(\omega) * G(\omega)$

$$\begin{aligned}
F(j\omega) &= \frac{1}{1+\omega^2} \\
G(j\omega) &= \frac{1}{1+\omega^2} \\
S_X(j\omega) &= F(j\omega) * G(j\omega) \\
\frac{1}{1+\omega^2} &\xrightarrow{\mathscr{F}^{-1}} \frac{1}{2}e^{-|t|} \\
S_X(j\omega) &\xrightarrow{\mathscr{F}^{-1}} \frac{1}{4}\int_{-\infty}^{\infty} e^{-|\tau|}e^{-|t-\tau|}d\tau
\end{aligned}$$

If $t \geq 0$,

$$\begin{aligned}
S_X(j\omega) &\xrightarrow{\mathscr{F}^{-1}} \int_{-\infty}^{0} e^{\tau}e^{-t+\tau}d\tau + \int_{0}^{t} e^{-\tau}e^{-t+\tau}d\tau + \int_{t}^{\infty} e^{-\tau}e^{-\tau+t}d\tau \\
&\xrightarrow{\mathscr{F}^{-1}} \frac{1}{2}[e^0 - e^{-\infty}] + te^{-t} + -\frac{1}{2}[e^{-\infty} - e^{-t}] \\
&\xrightarrow{\mathscr{F}^{-1}} te^{-t} + \frac{1}{2}e^{-t} + \frac{1}{2}
\end{aligned}$$

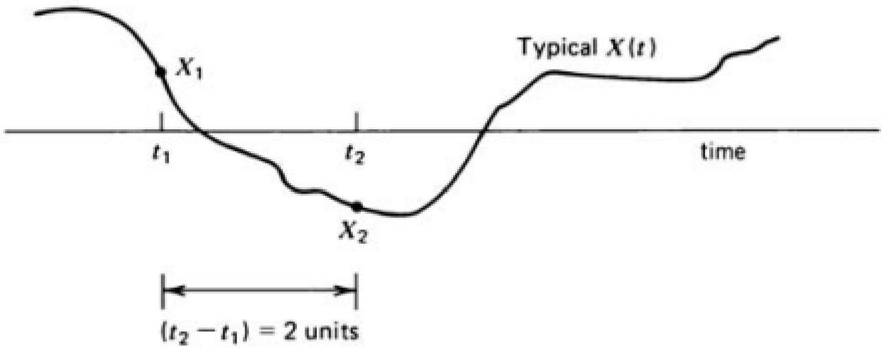

Figure P2.19

If $t < 0$,

$$\begin{aligned} S_X(j\omega) &\xrightarrow{\mathscr{F}^{-1}} \int_{-\infty}^{t} e^{\tau}e^{-t+\tau}d\tau + \int_{t}^{0} e^{\tau}e^{-\tau+t}d\tau + \int_{0}^{\infty} e^{-\tau}e^{-\tau+t}d\tau \\ &\xrightarrow{\mathscr{F}^{-1}} \frac{1}{2}[e^t - e^0] + te^t + -\frac{1}{2}[e^{-\infty} - e^0] \\ &\xrightarrow{\mathscr{F}^{-1}} -te^t + \frac{1}{2}e^t + \frac{1}{2} \end{aligned}$$

Combining both cases, we get $S_X(j\omega) \xrightarrow{\mathscr{F}^{-1}} |t|e^{-|t|} + \frac{1}{2}e^{-|t|} + \frac{1}{2}$.

2.17 A stationary process $X(t)$ is Gaussian and has an autocorrelation function of the form $R_X(\tau) = 4e^{-|\tau|}$. Let the random variable X_1 denote $X(t_1)$ and X_2 denote $X(t_1+1)$. Write the expression for the joint probability density function $f_{X_1X_2}(x_1, x_2)$.

First, note that the process mean is 0. The covariance matrix in this case is 2×2.

$$\begin{aligned} C_X &= \begin{bmatrix} E[X_1^2] & E[X_1X_2] \\ E[X_2X_1] & E[X_2^2] \end{bmatrix} = \begin{bmatrix} R_X(0) & R_X(1) \\ R_X(1) & R_X(0) \end{bmatrix} = \begin{bmatrix} 4 & 4e^{-1} \\ 4e^{-1} & 4 \end{bmatrix} \\ \mathbf{x} &= [x_1, x_2]^T \\ f_{X_1X_2}(\mathbf{x}) &= \frac{1}{2\pi|C_X|^{1/2}} e^{-\frac{1}{2}*\mathbf{x}^T*C_X^{-1}*\mathbf{x}} \end{aligned}$$

2.18 A stationary Gaussian process $X(t)$ has a power spectral density function $S_X(j\omega) = \frac{2}{\omega^4+1}$. Find $E[X]$ and $E[X^2]$.

From equation 2.9.7 (2nd order Gauss Markov process), $S_X(jw) = \frac{2\sqrt{2}\omega_0^3\sigma^2}{\omega^4+\omega_0^4}$, will give us equation 2.9.8 $R_X(\tau) = \sigma^2 e^{-a\tau}(cosa\tau + sina\tau)$ for $\tau \geq 0$ where $a = \frac{\omega_0}{\sqrt{2}}$. For our problem, we have $\omega_0 = 1$ and $\sigma^2 = \frac{1}{\sqrt{2}}$. Given the nature of Gauss-Markov processes, we have $E[X] = 0$, while $E[X^2] = R_X(0) = \frac{1}{\sqrt{2}}e^{-1/\sqrt{2}*0}(cos0 + sin0) = \frac{1}{\sqrt{2}}$.

2.19 A typical sample function of a stationary Gauss–Markov process is shown in the sketch. The process has a mean-square value of 9 units, and the random variables X_1 and X_2 indicated on the waveform have a correlation coefficient of 0.5. Write the expression for the autocorrelation function of $X(t)$.

First, note that the process mean is 0. The covariance matrix in this case is 2×2.

$$C_X = \begin{bmatrix} E[X_1^2] & E[X_1X_2] \\ E[X_2X_1] & E[X_2^2] \end{bmatrix} = \begin{bmatrix} R_X(0) & R_X(2) \\ R_X(2) & R_X(0) \end{bmatrix} = \begin{bmatrix} 9 & 9 * 0.5 \\ 9 * 0.5 & 9 \end{bmatrix}$$

$$\mathbf{x} = [x_1, x_2]^T$$

$$f_{X_1X_2}(\mathbf{x}) = \frac{1}{2\pi|C_X|^{1/2}} e^{-\frac{1}{2} * \mathbf{x}^T * C_X^{-1} * \mathbf{x}} = \frac{1}{15.58\pi} e^{-\frac{2*(x_1^2 - x_1x_2 + x_2^2)}{27}}$$

2.20 It was stated in Section 2.9 that a first-order Gauss–Markov process has an autocorrelation function of the form $R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$. It was also stated that a discrete-time version of the process can be generated by the recursive equation $X(t_{k+1}) = e^{-\beta\Delta t}X(t_k) + W(t_k)$ where $W(t_k)$ is a zero-mean white Gaussian sequence that is uncorrelated with $X(t_k)$ and all the preceding X samples. Show that $E[W^2(t_k)] = \sigma^2(1 - e^{-2\beta\Delta t})$. Note that in simulating a first-order Gauss–Markov process, the initial $X(t_0)$ must be a $N(0, \sigma^2)$ sample in order for the process to be stationary.

Note $E[X(t_{k+1})^2] = E[X(t_k)^2] = \sigma^2$ and $E[X(t_{k+1})X(t_k)] = R_X(\Delta t)$.

$$\begin{aligned} W(t_k) &= X(t_{k+1}) - e^{-\beta\Delta t}X(t_k) \\ E[W(t_k)^2] &= E[(X(t_{k+1}) - e^{-\beta\Delta t}X(t_k))^2] \\ &= E[X(t_{k+1})^2] - 2e^{-\beta\Delta t}E[X(t_{k+1})X(t_k)] + e^{-2\beta\Delta t}E[X(t_k)^2] \\ &= \sigma^2 - 2e^{-\beta\Delta t}\sigma^2 e^{-\beta\Delta t} + e^{-2\beta\Delta t}\sigma^2 \\ &= \sigma^2(1 - e^{-2\beta\Delta t}) \end{aligned}$$

2.21 In Example 2.12, it was mentioned that the derivative of a first-order Gauss–Markov process does not exist, and this is certainly true in the continuous-time case. Yet, when we look at adjacent samples of a typical discrete-time first-order process (see Fig. 2.16a), it appears that the difference between samples is modest and well behaved. Furthermore, we know that the $X(t_k)$ samples evolve in accordance with the recursive equation $X(t_{k+1}) = e^{-\beta\Delta t}X(t_k) + W(t_k)$ where $W(t_k)$ is an uncorrelated random sequence with a variance $\sigma^2(1 - e^{-2\beta\Delta t})$. Thus, as Δt becomes smaller and smaller, the variance of $W(t_k)$ approaches zero. Thus, it appears that the $X(t_k)$ sequence becomes "smooth" as Δt approaches zero, and one would think intuitively that the derivative (i.e., slope) would exist. This, however, is a mirage, because in forming the slope as the ratio $\frac{X(t_{k+1}) - X(t_k)}{\Delta t}$, both numerator and denominator approach zero as $\Delta t \to 0$. Show that the denominator in the approximate slope expression approaches zero "faster" than the numerator, with the result that the approximate slope becomes larger as Δt becomes smaller. This confirms that the approximate derivative does not converge as $\Delta t \to 0$, even in the discrete time case.

Note that $W(t_k)$ is zero mean and as $\Delta t \to 0$, its variance also goes to 0, meaning that $W(t_k) = 0$ as $\Delta t \to 0$. Hence we have the following equation:

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{X(t_{k+1}) - X(t_k)}{\Delta t} &= \frac{e^{-\beta\Delta t}X(t_k) + W(t_k) - X(t_k)}{\Delta t} \\ &= \frac{(e^0 - 1)X(t_k)}{0} \end{aligned}$$

2.22 We wish to determine the autocorrelation function a random signal empirically from a single time record. Let us say we have good reason to believe the process is ergodic and at least approximately Gaussian and,furthermore, that the autocorrelation function of the process decays exponentially with a time constant no greater than 10s. Estimate the record length needed to achieve 5 percent accuracy in the determination of the autocorrelation function. (By 5 percent accuracy, assume we mean that for any t, the standard deviation of the experimentally determined autocorrelation function will not be more than 5 percent of the maximum value of the true autocorrelation function.)

$$1/\beta = 10s$$
$$\frac{VarV(\tau)}{\sigma^2} \leq \frac{2}{\beta T}$$
$$T = \frac{2*10}{0.05^2} = 8000s$$

2.25 If you read the book, this is a matlab exercise. Skip for now.

3 Chapter 3

3.1 Lecture 14 - 16 (3.1 - 3.11)

3.1 Find the steady-state mean-square value of the output for the following filters. The input is white noise with a spectral density amplitude A.

(a) $G(s) = \frac{Ts}{(1+Ts)^2}$

$$c(s) = \sqrt{A}Ts$$
$$d(s) = T^2s^2 + 2Ts + 1$$
$$E[X^2] = \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} = \frac{(\sqrt{A}T)^2 + 0^2 * T^2}{2*1*2T*T^2} = \frac{A}{4T}$$

(b) $G(s) = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$

$$c(s) = \sqrt{A}\omega_0^2$$
$$d(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$
$$E[X^2] = \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} = \frac{0 + (\sqrt{A}\omega_0^2)^2 * 1}{2*\omega_0^2 * 2\zeta\omega_0 * 1} = \frac{A\omega_0}{4\zeta}$$

(c) $G(s) = \frac{s+1}{(s+2)^2}$

$$c(s) = \sqrt{A} * (s+1)$$
$$d(s) = s^2 + 4s + 4$$
$$E[X^2] = \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} = \frac{(\sqrt{A})^2 * 4 + (\sqrt{A})^2 * 1}{2*4*4*1} = \frac{5A}{32}$$

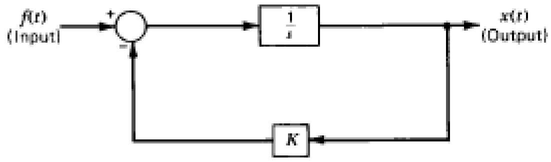

Figure P3.3

3.3 The input to the feedback system shown is a stationary Markov process with an autocorrelation function $R_f(\tau) = \sigma^2 e^{-\beta|\tau|}$. The system is in a stationary condition.

(a) What is the spectral density function of the output? $\frac{x(t)}{f(t)} = \frac{1}{s+K}$ (b) What is the mean square value of the output?

$$\begin{aligned} \mathscr{F}[R_f(\tau)] &= \frac{2\sigma^2\beta}{-s^2+\beta^2} \\ c(s) &= \sqrt{2\sigma^2\beta} * 1 \\ d(s) &= (s+\beta)*(s+K) = s^2 + (K+\beta)s + \beta K \\ E[X^2] &= \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} = \frac{0 + (\sqrt{2\sigma^2\beta} * 1)^2 * 1}{2 * \beta K * (K+\beta) * 1} = \frac{\sigma^2}{K(K+\beta)} \end{aligned}$$

3.4 Find the steady-state mean-square value of the output for a first-order low-pass filter, i.e., $G(s) = 1/(1+Ts)$ if the input has an autocorrelation function of the form

$$R(\tau) = \begin{cases} \sigma^2(1-\beta|\tau|), & -\frac{1}{\beta} \le \tau \le \frac{1}{\beta} \\ 0, & \textbf{otherwise} \end{cases}$$

Hint: The input spectral function is irrational so the integrals given in Table 3.1 are of no help here. One approach is to write the integral expression for $E(X^2)$ in terms of real ω rather than s and then use conventional integral tables. Also, those familiar with residue theory will find that the integral can be evaluated by the method of residues.

$$\begin{aligned} S_f(j\omega) = \mathscr{F}[R(\tau)] &= \frac{2}{\beta}\left(\frac{sin(\omega/\beta)}{\omega/\beta}\right)^2 \\ S_x(j\omega) &= \frac{2}{\beta}\left(\frac{sin(\omega/\beta)}{\omega/\beta}\right)^2 * \frac{1}{1+(T\omega)^2} \\ E(x^2) &= \frac{1}{2\pi}\frac{2}{\beta}\int_{-\infty}^{\infty}\left(\frac{sin(\omega/\beta)}{\omega/\beta}\right)^2 * \frac{1}{1+(T\omega)^2} d\omega \end{aligned}$$

I am not sure how to integrate $E(x^2)$. Skip.

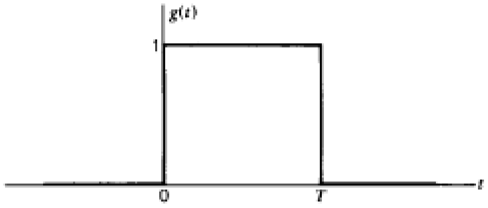

Figure P3.5 Filter weighting function.

3.5 Consider a linear filter whose weighting function is shown in the figure. (This filter is sometimes referred to as a finite-time integrator.) The input to the filter is white noise with a spectral density amplitude A, and the filter has been in operation a long time. What is the mean-square value of the output?

$$\begin{aligned}
\mathscr{L}[g(t)] &= \int_0^T e^{-st}dt = (1-e^{-sT}) * \frac{1}{s} \\
E(x^2) &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G(s) * G(-s) * A ds \\
&= \int_{-\infty}^{\infty} \frac{-A}{s^2}\left(2 - e^{sT} - e^{-sT}\right) ds \\
&= \frac{2A}{s}\Big|_{-\infty}^{\infty}
\end{aligned}$$

arg... skip this part again

3.6 Find the shaping filter that will shape unity white noise into noise with a spectral function $S(j\omega) = \frac{\omega^2+1}{\omega^4+8\omega^2+16}$.

$$\begin{aligned}
S(j\omega) &= G(j\omega) * G(-j\omega) * 1 \\
&= \frac{1+j\omega}{(j\omega)^2 - 4} * \frac{1-j\omega}{(-j\omega)^2-4} * 1 \\
G(j\omega) &= \frac{1+j\omega}{(j\omega)^2-4}
\end{aligned}$$

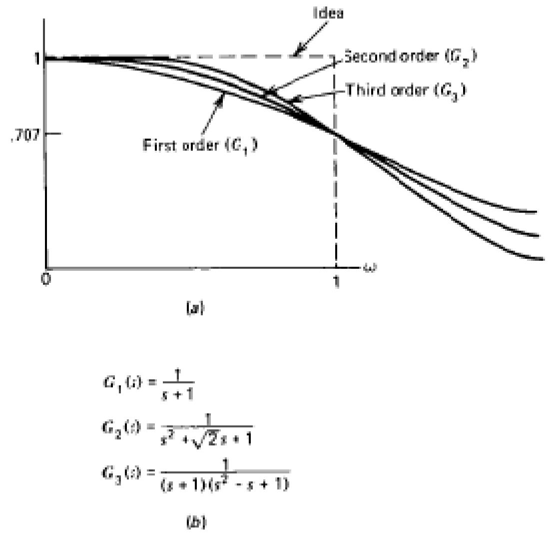

Figure P3.8 (*a*) Responses of three Butterworth filters.
(*b*) Transfer functions of Butterworth filters.

3.8 The transfer functions and corresponding bandpass characteristics for first-, second-, and third-order Butterworth filters are shown in the figure below. These filters are said to be "maximally flat" at zero frequency with each successive higher order filter more nearly approaching the ideal curve than the previous one. All three filters have been normalized such that all responses intersect the 3-db point at 1 rad/s (or $1/2\pi$ Hz).

(a) Find the noise equivalent bandwidth for each of the filters.

$$B = \frac{1}{2}[\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G(s)G(-s)ds]$$

$$c_1(s) = 1$$

$$d_1(s) = s + 1$$

$$B_1 = \frac{1}{2}\frac{c_0^2}{2d_0d_1} = \frac{1}{2}\frac{1^2}{2*1*1} = \frac{1}{4}$$

$$c_2(s) = 1$$

$$d_2(s) = s^2 + \sqrt{2}s + 1$$

$$B_2 = \frac{1}{2}\frac{c_1^2d_0 + c_0^2d_2}{2d_0d_1d_2} = \frac{1}{2}\frac{0 + 1^2 * 1}{2*1*\sqrt{2}*2} = \frac{1}{8\sqrt{2}}$$

$$c_3(s) = 1$$

$$d_3(s) = s^3 + 2s^2 + 2s + 1$$

$$B_3 = \frac{1}{2}\frac{c_2^2d_0d_1 + (c_1^2 - 2c_0c_2)d_0d_3 + c_0^2d_2d_3}{2d_0d_3(d_1d_2 - d_0d_3)} = \frac{0 + 0 + 1^2 * 1 * 2}{2*2*1*1*(4-1)} = \frac{1}{6}$$

(b) Insofar as noise suppression is concerned, is there much to be gained by using anything higher-order than a third-order Butterworth filter? Ideally, we have to use a filter with least "noise equivalent bandwidth". Since $B_2 < B_3$, there is no point on using higher order filters (I assuming that higher order filters won't give me lower B_k).

3.9 Find the mean-square value of the output (averaged in an ensemble sense) for the following transfer functions. In both cases, the initial conditions are zero and the input $f(t)$ is applied at $t = 0$.

(a) $G(s) = \frac{1}{s^2}, R_f(\tau) = A\delta(\tau)$

$$\begin{aligned} g(t) &= t \\ E[x^2(t)] &= \int_0^t \int_0^t uvA\delta(u-v)dudv \\ &= A\int_0^t v^2 dv = A\frac{t^3}{3} \end{aligned}$$

Note that as $t \to \infty$, the mean square value also goes to ∞.

(b) $G(s) = \frac{1}{s^2+\omega_0^2}, R_f(\tau) = A\delta(\tau)$

$$\begin{aligned} g(t) &= \frac{\sin\omega_0 t}{\omega_0} \\ E[x^2(t)] &= \int_0^t \int_0^t \frac{\sin\omega_0 u}{\omega_0}\frac{\sin\omega_0 v}{\omega_0} A\delta(u-v)dudv \\ &= \frac{A}{\omega_0^2}\int_0^t (\sin\omega_0 v)^2 dv = \frac{A}{\omega_0^2}\left(\frac{t}{2} - \frac{\sin 2\omega_0 t}{4\omega_0}\right) \end{aligned}$$

3.10 A certain linear system is known to satisfy the following differential equation:

$$\ddot{x} + 10\dot{x} + 100x = f(t)x(0) = \dot{x}(0) = 0$$

where $x(t)$ is the response (say, position) and the $f(t)$ is a white noise forcing function with a power spectral density of 10 units. (Assume the units are consistent throughout.)

(a) Let x and $\dot{x}$ be state variables x_1 and x_2. Then write out the vector state space differential equations for this random process.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -10 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

(b) Suppose this process is sampled at a uniform rate beginning at $t = 0$. The sampling interval is 0.2 s. Find the ϕ and Q parameters for this situation. (The Van Loan method described in Section 3.9 is recommended.)

(c) What are the mean-square values of $x(t)$ and $\dot{x}(t)$ at $t = 0.2$s?

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