Optimal State Estimation Kalman, H∞, and Nonlinear Approaches Solutions

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This is my personal attempt to the "digest" or "solve the end chapter problems in" the book "Simon, Dan. Optimal state estimation: Kalman, H infinity, and nonlinear approaches. John Wiley & Sons, 2006". This document is by no means a perfect solution manual for the said book, but was written simple for my personal enrichment.

1 Linear Systems Theory

1.4 Find the partial derivative of the trace of AB with respect to A.

$Tr(AB) = B^T$

1.5 Consider the matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Recall that the eigenvalues of A are found by find the roots of the polynomial $P(\lambda) = |\lambda I - A|$. Show that $P(A) = 0$.

$$\begin{aligned} P(\lambda) &= \begin{bmatrix} \lambda - a & b \\ b & \lambda - c \end{bmatrix} \\ &= \lambda^2 - (a+c)\lambda + ac - b^2 \\ P(A) &= A^2 - (a+c)A + (ac-b)^2 I \\ &= \begin{bmatrix} a^2+b^2 & ab+bc \\ ab+bc & b^2+c^2 \end{bmatrix} - \begin{bmatrix} a^2+ac & ab+bc \\ ab+bc & ac+c^2 \end{bmatrix} + \begin{bmatrix} ac-b^2 & 0 \\ 0 & ac-b^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

1.6 Suppose that A is invertible and $\begin{bmatrix} A & A \\ B & A \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, Find B and C in terms of A [Lie67].

$$A^2 + AC = 0 \tag{1.1}$$

$$AC = -A^2 \tag{1.2}$$

$$C = -A \tag{1.3}$$

$$BA + AC = I \tag{1.4}$$

$$BA - AA = I \tag{1.5}$$

$$B - A = A^-1 \tag{1.6}$$

$$B = A + A^-1 \tag{1.7}$$

1.8 Consider the matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where a, b, and c are real, and a and c are nonnegative.

a) Compute the solutions of the characteristic polynomial of A to prove that the eigenvalues of A are real.

$$
\begin{aligned}
P(\lambda) &= \lambda^2 - (a+c)\lambda + ac - b^2 \\
\lambda &= \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} \\
&= \frac{(a+c) \pm \sqrt{(a-c)^2 + b^2}}{2}
\end{aligned}
$$

The value inside sqrt will always be positive, therefore λ will always be real.

b) For what values of b is A positive semidefinite?

$$
\begin{aligned}
x^T A x &\geq 0 \\
x_1^2 a + 2x_1 x_2 b + x_2^2 c &\geq 0 \\
b &\geq \frac{-(x_1^2 a + x_2^2 c)}{2x_1 x_2}
\end{aligned}
$$

1.10 Suppose that the matrix A has eigenvalues λ_i, and eigenvectors v_i $(i = 1, \ldots, n)$. What are the eigenvalues and eigenvectors of $-A$?

A can be diagonalized as $S\Lambda S^{-1}$. Consequently, $-A$ can be expressed as $S(-\Lambda)S^{-1}$, where we can observe that $-A$ has the same eigenvectors as A, but with all its eigenvalues multiplied by -1.

1.11 Show that $|e^{At}| = e^{|A|t}$ for any square matrix A.

$$
\begin{aligned}
|e^{At}| &= |Q| * |e^{\hat{A}t}| * |Q^{-1}| \\
&= \begin{vmatrix} e^{\hat{A}_{11}t} & & \\ & \ddots & \\ & & e^{\hat{A}_{nn}t} \end{vmatrix} \\
&= e^{\hat{A}_{11}t + \cdots + \hat{A}_{nn}t} = e^{Tr(A)t}
\end{aligned}
$$

I can show this, but I can't show $|e^{At}| = e^{|A|t}$.

1.12 Show that if $\dot{A} = BA$, then $\frac{d|A|}{dt} = Tr(B)|A|$.

Let $A = F(t)$, with zero input, we can obtain $F(t) = e^{Bt}F(0)$.

$$
\begin{aligned}
|F(t)| &= |e^{Bt}||F(0)| \\
&= e^{Tr(B)t}|F(0)| \\
\frac{d|F(t)|}{dt} &= Tr(B)e^{Tr(B)t}|F(0)| \\
&= Tr(B)|F(t)| \\
\frac{d|A|}{dt} &= Tr(B)|A|
\end{aligned}
$$

1.13 The linear position p of an object under constant acceleration is $p = p_o + \dot{p}t + \frac{1}{2}\ddot{p}t^2$ where p_0 is the initial position of the object.

a) Define a state vector as $x = \begin{bmatrix} p & \dot{p} & \ddot{p} \end{bmatrix}^T$ and write the state space equation $\dot{x} = Ax$ for this system. $\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix}$

b) Use all three expressions in Equation (1.71) to find the state transition matrix e^{At} for the system. I'll only use the first one. Note that $A^3 = 0$.

$$\begin{aligned} e^{At} &= I + At/1! + A^2t^2/2! \\ &= \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

c) Prove for the state transition matrix found above that $e^{A0} = I$. Direct subtition from the obtain e^{At} above does give us I.

1.14 Consider the following system matrix. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Show that $S(t) = \begin{bmatrix} e^t & 0 \\ 0 & 2e^{-t} \end{bmatrix}$ satisfies the relation $\dot{S}(t) = AS(t)$, but $S(t)$ is not the state transition matrix of the system.

$$\begin{aligned} \dot{S}(t) &= \begin{bmatrix} e^t & 0 \\ 0 & -2e^{-t} \end{bmatrix} \\ &= A * S(t) \\ e^{At} &= Qe^{\hat{A}t}Q^-1 \\ &= I * \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} * I \end{aligned}$$

1.16 Show (H, F) is an observable matrix pair if and only if (H, F^{-1}) is observable (assuming that F is nonsingular).

Note: $Rank(AB) = min(Rank(A), Rank(B))$ Let the observability matrix of (H, F^{-1}) be G, then the observability matrix of (H, F) is GF^{n-1} (and some row swapping which should not affect rank). Assuming F must be rank n, then $Rank(GF^{n-1}) = min(n, n) = n$.