Probability Theory Problem Solving

2.1 0th moment $E(X^0) = E(1)$

$$= \int_{-\infty}^{\infty} f_x(x)\,dx = \boxed{1}$$

0th Central Moment $E((X-\bar{X})^0) = E(1)$

$$= \boxed{1}$$

2.2 $\frac{4!}{4^4} = \frac{\#\text{ of } \cancel{\text{unique}} \text{ ways where each pile has one ace}}{\#\text{ of ways the Ace can be distributed}}$

2.3 $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

$\int_0^1 ax(1-x)\,dx = 1$

$a\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 1$

$a\left(\frac{1}{2} - \frac{1}{3}\right) = 1 \Rightarrow \boxed{a = 6}$

2.4 $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

$\int_{-\infty}^{\infty} a\frac{1}{e^x + e^{-x}}\,dx = 2a\int_0^{\infty} \frac{1}{e^x + e^{-x}}\,dx$ because $\frac{1}{e^x+e^{-x}}$ is an even function

$= 2a\int_0^{\infty} \frac{e^x}{e^{2x}+1}\,dx = 2a\int_0^{\infty} \frac{1}{u^2+1}\,du$ where $u = e^x$, $du = e^x dx$

$= 2a \tan^{-1} u\Big|_{x=0}^{x=\infty} = 2a \tan^{-1} e^x\Big|_0^{\infty} = 2a\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$= \frac{a\pi}{2}$ equate this to 1

$\frac{a\pi}{2} = 1 \Rightarrow \boxed{a = \frac{2}{\pi}}$

What is the probability that $X \leq 1$?

$\int_{-\infty}^{1} f_x(x) = \int_{-\infty}^{1} \frac{2}{\pi}\frac{1}{e^x + e^{-x}}\,dx$ by taking note of the integration done above...

$= \frac{2}{\pi}\tan^{-1} e^x\Big|_{-\infty}^{1} = \frac{2}{\pi}\left[\tan^{-1} e - \tan^{-1} 0\right]$

$= \frac{2}{\pi}\tan^{-1} e \approx 0.7756$

2.5 Given probability density function $f_x(x) \begin{cases} ae^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$

where $a \geq 0$

a) Find the probability distribution function

$F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx$

For $x \leq 0$, $F_x(x) = 0$

For $x > 0$, $\int_0^x ae^{-ax}\,dx = -e^{-ax}\Big|_0^x$

$= 1 - e^{-ax}$

$F_x(x) \begin{cases} 1 - e^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$

b) Mean

Integration by parts: $\int u\,dv = uv - \int v\,du$

Mean $= \int_{-\infty}^{\infty} x f_x(x) = \int_0^{\infty} x\,ae^{-ax}\,dx$

let $u = x$ $du = dx$ $v = -\frac{e^{-ax}}{\cancel{a}}$ $dv = ae^{-ax}dx$

$= -xe^{-ax}\Big|_0^{\infty} - \int_0^{\infty} -e^{-ax}\,dx$

$= (0-0) - \frac{e^{-ax}}{a}\Big|_0^{\infty} = \boxed{\frac{1}{a}}$

c) ~~Formula~~ Second Moment

$E[x^2] = \int_0^{\infty} x^2 ae^{-ax}\,dx$

Note $\int x^2 e^{ax}\,dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$

$= a\left(\frac{x^2}{-a} - \frac{2x}{a^2} - \frac{2}{a^3}\right)e^{-ax}\Big|_0^{\infty} = a - \left(0 - 0 - \frac{2}{a^3}\right)$

$= \boxed{\frac{2}{a^2}}$

d) Variance $\sigma_x^2 = E[x^2] - (E[x])^2 = \frac{2}{a^2} - \frac{1}{a^2} = \boxed{\frac{1}{a^2}}$

e) Prob w/in $1\ \sigma_x = \frac{1}{a}$ from mean $\frac{1}{a}$

$\int_{\frac{1}{a}-\frac{1}{a}}^{\frac{1}{a}+\frac{1}{a}} ae^{-ax}\,dx = -e^{-ax}\Big|_0^{\frac{2}{a}} = \boxed{1 - e^{-2} = 0.8647}$

2.6 express skewness in terms of 1st, 2nd, 3rd moments

$$E\left[(X-\bar{X})^3\right] = E\left[X^3 + 3X^2\bar{X} + 3X\bar{X}^2 + \bar{X}^3\right]$$

$$= E[X^3] + 3E[X^2]E[X] + 3E[X]E[X]^2 + E[X]^3 = \boxed{E[X^3] + 3E[X^2]E[X] + 4E[X]^3}$$

2.7 Given pdf $f_X(x) = \frac{ab}{b^2+x^2}$ where $b>0$

a) $\int_{-\infty}^{\infty} \frac{ab}{b^2+x^2}\,dx = \int_0^{\infty} \frac{2ab}{b^2+x^2}\,dx$ since $f_X(x)$ here is even func.

Note. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$= 2ab\,\frac{1}{b}\tan^{-1}\frac{x}{b}\Big|_0^{\infty} = 2a\frac{\pi}{2}$ equating to 1 gives us $\boxed{a = \frac{1}{\pi}}$

b) Find the mean

$\int_{-\infty}^{\infty} x\,\frac{b}{\pi(b^2+x^2)}\,dx$ notice however that this func is odd, $(f(-x) = -f(x))$

$\therefore$ mean $= \boxed{0}$

2.8 what is the std of $X = W+V$ given W and V are uncorrelated

$$E[(X-\bar{X})^2] = E[X^2] - E[X]^2 = E[W^2] + E[V^2] - (E[W]+E[V])^2$$

$$= E[W^2] - E[W]^2 + E[V^2] - E[V]^2 + 2E[W]E[V]$$

$$\boxed{\sigma_X^2 = \sigma_W^2 + \sigma_V^2 + 2\mu_W\mu_V}$$

2.9 Given RV X and Y

a) Prove if X and Y are independent, $\rho = 0$ (corr. coeff)

$\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{E[XY] - \bar{X}\bar{Y}}{\sigma_X\sigma_Y}$ If X and Y are independent $E[XY] = \bar{X}\bar{Y}$

$$= \frac{\bar{X}\bar{Y} - \bar{X}\bar{Y}}{\sigma_X\sigma_Y} = 0$$

b) Let $X =$ uniform $(-1, 1)$, $Y = X^2$ $\quad \rho = \frac{0-0}{\sigma_X\sigma_Y} = 0$

c) Prove that if $Y = aX+b$, then $\rho = \pm 1$

- $\bar{y} = a\bar{x} + b$
- $E[(y-\bar{y})^2] = \sigma_y^2 = E[y^2] - \bar{y}^2 = E[(ax+b)^2] - (a\bar{x}+b)^2$

$$= a^2E[x^2] + 2ab\bar{x} + b^2 - (a^2\bar{x}^2 + 2ab\bar{x} + b^2)$$

$$= a^2\sigma_x^2 \Rightarrow \sigma_y = \pm a\sigma_x$$

- $\rho = \frac{E[XY] - \bar{x}\bar{y}}{\sigma_x\sigma_y} = \frac{E[x(ax+b)] - \bar{x}(a\bar{x}+b)}{\sigma_x\sigma_y} = \frac{a\sigma_x^2}{\pm a\sigma_x\sigma_x} = \pm 1$

2.10 Given $f_{XY}(x,y) = \begin{cases} ae^{-2x}e^{-3y} & x>0\ y>0 \\ 0 & \text{otherwise} \end{cases}$

a) Find a. $\int_0^{\infty}\int_0^{\infty} ae^{-2x}e^{-3y}\,dx\,dy$ must be equal to 1

$$= a\left(\frac{e^{-2x}}{-2}\Big|_0^{\infty}\right)\left(\frac{e^{-3y}}{-3}\Big|_0^{\infty}\right) = a\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \Rightarrow \boxed{a=6}$$

b) $E[X] = \int_0^{\infty}\int_0^{\infty} x\,6e^{-2x}e^{-3y}\,dx\,dy$ note: $\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$

$$= 6\left(\frac{x}{-2} - \frac{1}{(-2)^2}\right)e^{-2x}\Big|_0^{\infty}\left(\frac{e^{-3y}}{-3}\Big|_0^{\infty}\right)$$

$$= 6\left(0 - \left(0 - \frac{1}{4}\right)\right)\left(\frac{1}{3}\right) = \boxed{\frac{1}{2}}$$

$$E[Y] = \int_0^{\infty}\int_0^{\infty} y\,6e^{-2x}e^{-3y}\,dx\,dy$$

$$= 6\left(\frac{y}{-3} - \frac{1}{(-3)^2}\right)e^{-3y}\Big|_0^{\infty}\left(\frac{e^{-2x}}{-2}\right)\Big|_0^{\infty}$$

$$= 6\left(0 - \left(0 - \frac{1}{9}\right)\right)\left(\frac{1}{2}\right) = \boxed{\frac{1}{3}}$$

c) $E[X^2] = \int_0^{\infty}\int_0^{\infty} x^2\,6e^{-2x}e^{-3y}\,dx\,dy$ note: $\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$

$$= 6\left(\frac{x^2}{-2} - \frac{2x}{(-2)^2} + \frac{2}{(-2)^3}\right)e^{-2x}\Big|_0^{\infty}\left(\frac{e^{-3y}}{-3}\right)\Big|_0^{\infty}$$

$$= 6\left(0 - \left(0 - 0 - \frac{2}{8}\right)\right)\left(\frac{1}{3}\right) = \boxed{\frac{1}{2}}$$

$$E[Y^2] = \int_0^{\infty}\int_0^{\infty} y^2\,6e^{-2x}e^{-3y}\,dx\,dy$$

$$= 6\left(\frac{y^2}{-3} - \frac{2y}{(-3)^2} + \frac{2}{(-3)^3}\right)e^{-3y}\Big|_0^{\infty}\left(\frac{e^{-2x}}{-2}\right)\Big|_0^{\infty}$$

$$= 6\left(0 - \left(0 - 0 + \frac{2}{3^3}\right)\right)\left(\frac{1}{2}\right) = \boxed{\frac{2}{9}}$$

$$E[XY] = \int_0^{\infty}\int_0^{\infty} xy\,6e^{-2x}e^{-3y}\,dx\,dy$$

$$= 6\left[\left(\frac{x}{-2} - \frac{1}{(-2)^2}\right)e^{-2x}\right]_0^{\infty}\left[\left(\frac{y}{-3} - \frac{1}{(-3)^2}\right)e^{-3y}\right]_0^{\infty}$$

$$= 6\left[0 - \left(0 - \frac{1}{4}\right)\right]\left[0 - \left(0 - \frac{1}{9}\right)\right] = \boxed{\frac{1}{6}}$$

d) $R = E\left[\begin{bmatrix}X\\Y\end{bmatrix}\begin{bmatrix}X & Y\end{bmatrix}\right] = \begin{bmatrix}E[X^2] & E[XY]\\ E[XY] & E[Y^2]\end{bmatrix} = \boxed{\begin{bmatrix}\frac{1}{2} & \frac{1}{6}\\ \frac{1}{6} & \frac{2}{9}\end{bmatrix}}$

e) $\sigma_X^2 = E[X^2] - E[X]^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$ $\quad C_{XY} = E[XY] - \bar{X}\bar{Y} = \frac{1}{6} - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \boxed{0}$

$\sigma_Y^2 = E[Y^2] - E[Y]^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \boxed{\frac{1}{9}}$

f) $C = R - \begin{bmatrix}\bar{X}\\\bar{Y}\end{bmatrix}\begin{bmatrix}\bar{X} & \bar{Y}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{6}\\ \frac{1}{6} & \frac{2}{9}\end{bmatrix} - \begin{bmatrix}\frac{1}{4} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{9}\end{bmatrix} = \boxed{\begin{bmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{9}\end{bmatrix}}$

g) $\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \boxed{0}$

2.11 $R_X(\tau) = Ae^{-k|\tau|}$

a) Power Spectrum $S_X(\omega)$

note: $F\left[e^{-a|t|}\right] = \cancel{\frac{1}{\sqrt{2\pi}}} \frac{2a}{a^2+\omega^2}$

$$S_X(\omega) = F\left[R_X(\tau)\right] = A\,\frac{2(\cancel{+}k)}{(\cancel{+}k)^2+\omega^2} = \boxed{\frac{2AK}{K^2+\omega^2}}$$

b) $P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2AK}{K^2+\omega^2}\,d\omega$ Note: $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$= \cancel{\frac{2AK}{2\pi}}\ \cancel{\frac{2}{K}}\tan^{-1}\frac{x}{K}\Big|_0^{\infty} = \frac{2A}{\pi}\left(\frac{\pi}{2}-0\right)$

$\boxed{P_X = A}$

c) $P_X = \frac{A}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2AK}{K^2+\omega^2}d\omega = \frac{2A}{\pi}\left(\tan^{-1}\frac{x}{K}\Big|_{-\infty}^{2\pi}\right)$

$\frac{A}{2} = \frac{2A}{\pi}\left(\tan^{-1}\frac{2\pi}{K} + \frac{\pi}{2}\right)$

$\tan^{-1}\frac{2\pi}{K} = -\frac{\pi}{4} \Rightarrow \frac{2\pi}{K} = -1 \Rightarrow \boxed{K = -2\pi}$

2.12 X is RV ; $Y(t) = X\cos t$

let $X = f(x)$

a) $E[Y(t)] = \int_{-\infty}^{\infty} x f(x)\cos t\,dx$

$\boxed{= \cos t\ \bar{X}}$

b) $A[Y(t)] = \lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\cos t\,dt$

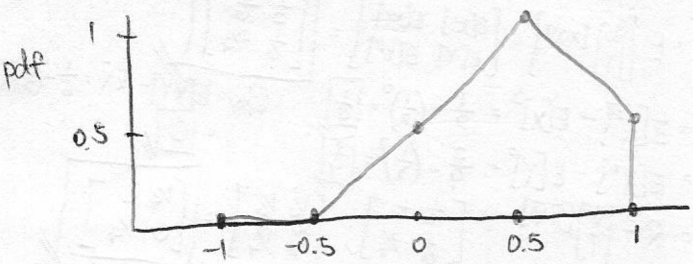
$= f(x)\lim_{t\to\infty}\frac{2\sin T}{T} = \boxed{0}$

c) $E[Y(t)] = A[Y(t)]$ if $Y(t)$ is ergodic

eg. $\boxed{t = \frac{\pi}{2} + n\pi}$

2.13 $Z = X + V$

a) Plot $p(Z=0.5\,|\,X)$



b) Conditional Expectation = mean of pdf above

Note: $f(x)\begin{cases} a(x+0.5) & x\in[-0.5, 0.5] \\ a(1.5-x) & x\in[0.5, 1]\end{cases}$

solve for a: $\int_{-\infty}^{\infty} f(x) = 1 = \int_{-0.5}^{0.5} a(x+0.5)dx + \int_{0.5}^{1} a(1.5-x)dx$

$= a\left(\frac{x^2}{2}\Big|_{-0.5}^{0.5} + 0.5x\Big|_{-0.5}^{0.5} + 1.5x\Big|_{0.5}^{1} - \frac{x^2}{2}\Big|_{0.5}^{1}\right)$

$= a\left(0 + 0.5 + 0.75 - 0.375\right) = a(0.875)$

$\boxed{\Rightarrow a \approx 1.14286}$ or $\frac{8}{7}$

$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = a\left(\int_{-0.5}^{0.5} x(x+0.5)dx + \int_{0.5}^{1} x(1.5-x)dx\right)$

$= a\left(\frac{x^3}{3}\Big|_{-0.5}^{0.5} + \frac{1}{2}\frac{x^2}{2}\Big|_{-0.5}^{0.5} + 1.5\frac{x^2}{2}\Big|_{0.5}^{1} - \frac{x^3}{3}\Big|_{0.5}^{1}\right)$

$= a\left(\frac{1}{12} + 0 + \frac{9}{16} - \frac{7}{24}\right) = \boxed{\frac{17}{42} \text{ or } 0.405}$

$\boxed{\text{Most Probable value of } X = 0.5}$

Median $\int_{-\infty}^{m} f(x)dx = 0.5$

$0.5 = a\left(\int_{-0.5}^{0.5}(x+0.5)dx + \int_{0.5}^{m}(1.5-x)dx\right)$

$= a\left(\frac{x^2}{2}\Big|_{-0.5}^{0.5} + 0.5x\Big|_{-0.5}^{0.5} + 1.5x\Big|_{0.5}^{m} - \frac{x^2}{2}\Big|_{0.5}^{m}\right)$

$0.5 = a\left(0 + 0.5 + 1.5m - 0.75 - \frac{m^2}{2} + \frac{1}{8}\right)$

$0 = -\frac{1}{2}m^2 + 1.5m - \frac{9}{16}$

$m = 2.56$, $\boxed{0.43934}$

impossible

P.S. my sol'n seem complicated so I am not confident w/ it.

2.14 No it is not ergodic. → temp at noon in London.

Temp. ~~do~~ range changes w/ season so 1 sample at summer isn't able to represent the temp. at winter, fall, spring.