ESE Chapter 3

3.1 Show $\frac{\partial^2 J}{\partial \hat{x}^2}$ is positive definite

$$\frac{\partial J}{\partial \hat{x}} = -2y^T H + 2\hat{x}^T H^T H$$

$$\frac{\partial^2 J}{\partial \hat{x}^2} = 0 + 2H^T H$$

and $H^T H$ is always positive!

3.2 Prove P_k is positive definite given P_{k-1} & R_k are positive definite

$$P_k = (I - K_k H_k) P_{k-1} (I - K_k H_k)^T + K_k R_k K_k^T$$

Given P_{k-1} is positive definite, then $\cancel{x^T A x > 0}$ $x^T P_{k-1} x > 0$ for all x ($n\times1$ vector), including $x' = x(I - K_k H_k)^T$. Same logic w/ R_k. If $(I - K_k H_k) P_{k-1} (I - K_k H_k)^T$ and $K_k R_k K_k^T$ is positive definite, then so does its sum.

3.3 $P_0 = \infty I$, $H_k = 1$.

$$K_1 = P_0 H_k^T (H_k P_0 H_k^T + R_k)^{-1}$$
$$= \infty 1 (\infty + R_k)^{-1} = \boxed{1}$$
$$P_1 = (I - K_k H_k) P_0 = (1 - 1*1)\infty = \boxed{0}$$

3.4

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

~~a) $\hat{x} = (H^T H)^{-1} H^T y = \frac{y_1 + y_2}{2}$~~

a) $$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y$$
$$= \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1} y$$
$$= \left(1 + \tfrac{1}{4}\right)^{-1} \left(\frac{y_1}{1} + \frac{y_2}{4} \right) = \frac{4}{5}\left(y_1 + \frac{y_2}{4} \right)$$
$$\boxed{= \frac{4}{5} y_1 + \frac{1}{5} y_2}$$

b) $\sigma_{\hat{x}}^2 = \frac{4}{5}(1) + \frac{1}{5}(4) = 1.6$ I am not ~~really~~ sure...

~~$\sigma_{\hat{x}}^2 = 1.6$?~~

~~$P_0 = \infty I$, $K_1 = \ldots 0$~~

c) ~~$\sigma = \frac{1}{2}(1) + \frac{1}{2}(4) = 2.5$~~

this might be correct ...

$K_k = 1\times2$

~~b) $P_0 = \infty$~~

~~$K_k = P_{k-1} H_k^T (H_k P_{k-1} H_k^T + R_k)^{-1}$~~

~~$K_1 = \frac{P_0 H_k}{P_0 \begin{bmatrix}1&1\end{bmatrix}\ldots} = \begin{bmatrix}1 & 1\end{bmatrix}$~~

~~$P_1 = \left(\frac{1}{\infty} + \begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)^{-1}$~~

~~$P_1 = \frac{4}{5}$~~ ~~$\infty 0 + \begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 5$~~

~~$K_2 = \ldots 5\begin{bmatrix}1&1\end{bmatrix}\left(\begin{bmatrix}5&5\\5&5\end{bmatrix} + \begin{bmatrix}1&0\\0&4\end{bmatrix}\right)^{-1} = \begin{bmatrix}20/29 & 5/29\end{bmatrix}$~~

~~$P_2 = 20/29$~~

~~this may actually be wrong as this is for recursive least squares~~

$K_k = \infty 1(\infty+1)^{-1} = 1$

$P_1 = 0 + 1$

$\boxed{P_1 = 1}$

$K_2 = 1 * (1+4)^{-1} = 1/5$

$P_2 = \left(\frac{4}{5}\right)^2 1 + \left(\frac{1}{5}\right)^2 4$

$\boxed{P_2 = \frac{4}{5}}$

c) ~~$R_k = \ldots$~~ ~~$K_1 = [1, 1]$~~ ~~not sure...~~ ~~Same as P_1 since $\hat{K}_1 = [1\ 1]$ force~~

$$E\left(\left(\frac{y_1 + y_2}{2} - \bar{x}\right)^2\right) = E\left(\left(\frac{v_1}{2}\right)^2\right) + E\left(\left(\frac{v_2}{2}\right)^2\right) = \frac{1}{4} + \frac{4}{4} = \boxed{5/4}$$

3.5 ~~$P_0 = 1$, $R = \begin{bmatrix}1&0\\0&4\end{bmatrix}$~~

$P_0 = 1$

$K_1 = \frac{1}{1+1} = \frac{1}{2}$

$P_1 = \frac{1}{2}^2 1 + \frac{1}{2}^2 1$

$\boxed{P_1 = 1/2}$

$K_2 = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2} + 4\right)} = 1/9$

$P_2 = \left(\frac{8}{9}\right)^2 \frac{1}{2} + \left(\frac{1}{9}\right)^2 4$

$\boxed{P_2 = 4/9}$

~~$\hat{x}_k = \hat{x}_{k-1} + K_k \left(\begin{bmatrix} y_{1} \\ y_{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \hat{x}_{k-1} \right)$~~

~~$K_k = P_{k-1}\begin{bmatrix}1&1\end{bmatrix}\left(\begin{bmatrix}1\\1\end{bmatrix} P_{k-1} \begin{bmatrix}1&1\end{bmatrix} + \begin{bmatrix}1&0\\0&4\end{bmatrix}\right)^{-1}$~~

~~$K_k = \begin{bmatrix}P_{k-1} & P_{k-1}\end{bmatrix}\begin{pmatrix}P_{k-1}+1 & P_{k-1} \\ P_{k-1} & P_{k-1}+4\end{pmatrix}^{-1}$~~

~~$K_1 = \begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}2&1\\1&5\end{bmatrix}^{-1} = \begin{bmatrix}4/9 & 1/9\end{bmatrix}$~~

~~$x_1 = 4/9 x_0 + 4/9 y_1 + 1/9 y_2$~~

~~$P_1 = 4/9$~~

~~$K_2 = \begin{bmatrix}0.2857 & 0.0714\end{bmatrix}$~~

~~$x_2 = 0.6429 x_0 + 0.2857 y_1 + 0.0714 y_2$~~

~~$P_2 = 0.2857$~~

3.6 a) $E(\hat{\bar{x}}) = \bar{x}$?

$$E(\hat{\bar{x}}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n} n \bar{x} = \bar{x}$$

b) Find $E(x_i x_j)$ in terms of $\bar{x}$ and σ^2

If $i \neq j$, $E[(x_i - \bar{x})(x_j - \bar{x})] = 0 = E[x_i x_j - \bar{x}x_i - x_j\bar{x} + \bar{x}^2]$

$\Rightarrow E[x_i x_j] - \bar{x}^2 = 0 \Rightarrow \boxed{E[x_i x_j] = \bar{x}^2}$

If $i = j$, $\sigma^2 = E[x^2] - \bar{x}^2 \Rightarrow \boxed{E[x_i x_j] = \bar{x}^2 + \sigma^2}$

c) $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\bar{x}})^2 \Rightarrow E(\hat{\sigma}^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\bar{x}})^2\right)$

$$= E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i^2 - x_i\hat{\bar{x}} + \hat{\bar{x}}^2)\right)$$

~~Not really sure...~~ $= \frac{1}{n}\left[\sum_{i=1}^{n} E(x_i^2) - \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E(x_i x_j) + \frac{1}{n^2}\sum_{i,j,k=1}^{n} E(x_j x_k)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} E(x_i^2) - \frac{1}{n}\sum_{i,j=1}^{n} E(x_i x_j)\right] = \frac{1}{n}\left[n(\bar{x}^2 + \sigma^2) - \frac{1}{n}\left[(n^2 - n)\bar{x}^2 + n(\bar{x}^2 + \sigma^2)\right]\right]$$

$$= \bar{x}^2 + \sigma^2 - \bar{x}^2 - \frac{1}{n}\sigma^2 = \boxed{\frac{n-1}{n}\sigma^2}$$

3.7

actual	1	2	3	RMSE	Absolute Error	std	
est 1	3	4	1	2	2	1.88	
est 2	1	2	6	1.73	1	1.41	best rms
est 3	5	6	7	4	4	0	best std

est 2 best intuitive est3 worst intuitive

3.8 $f(x) = \begin{cases} \frac{x}{0.75} & x \in [0, 0.75] \\ \frac{2}{1.25} - \frac{x}{1.25} & x \in [0.75, 2] \end{cases}$

$$\int_0^{0.75} \frac{x}{0.75} = \frac{1}{0.75} \frac{x^2}{2}\Big|_0^{0.75} = 0.375$$

$$\int_{0.75}^{2} \frac{2}{1.25} - \frac{x}{1.25} = \frac{2}{1.25}x\Big|_{0.75}^{2} - \frac{1}{1.25}\frac{x^2}{2}\Big|_{0.75}^{2} = \cancel{1.3125}\ 0.625$$

$$\int_{\hat{x}}^{\infty} \frac{2}{1.25} - \frac{x}{1.25}\,dx = 0.\cancel{84375}\ 0.5$$

$$\frac{2}{1.25}x\Big|_{\hat{x}}^{2} - \frac{1}{1.25}\frac{x^2}{2}\Big|_{\hat{x}}^{2} = 0.\cancel{84375}5 = \frac{2}{1.25}2 - \frac{2}{1.25}\hat{x} - \frac{2}{1.25} + \frac{\hat{x}^2}{2 \cdot 1.25}$$

$$\Rightarrow \boxed{\hat{x} = 0.882\ \cancel{0.5476}},\ \cancel{3.4524}\ 3.12$$

b) mode $\hat{x} = 0.75$

c) mean $= \int_0^{0.75} \frac{x^2}{0.75}dx + \int_{0.75}^{2} x\left(\frac{2}{1.25} - \frac{x}{1.25}\right)dx$

$$= \frac{x^3}{3 * 0.75}\Big|_0^{0.75} + \frac{2}{1.25}\frac{x^2}{2}\Big|_{0.75}^{2} - \frac{1}{1.25}\frac{x^3}{3}\Big|_{0.75}^{2} = \boxed{\frac{11}{12}}$$

d) $\hat{x} = 1$ ~~0 or 2~~ (not really sure why)

3.9 $\sigma_1^2 = 10$

a) combine $\sigma_1^2 = 10$ w/ $\sigma_2^2 = 6$

b) combine $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 10$

a) $P_0 = \infty$

$K_1 = \frac{P_0}{P_0 + R_0} = \frac{\infty}{\infty + 10} = 1$

$P_1 = (1 - K_1)^2 P_0 + K_1^2 R_0$
$= 10$

$K_2 = \frac{P_1}{P_1 + R_1} = \frac{5}{8}$

$P_2 = (1 - K_2)^2 P_1 + K_2^2 R_1$
$= 3.75$

b) $P_0 = \infty$

$K_1 = 1$

$P_1 = 10$

$K_2 = \frac{1}{2}$

$P_2 = 5$

$K_3 = \frac{P_2}{P_2 + R_2} = 1/3$

$P_3 = (1 - K_3)^2 P_2 + K_3^2 R_2$
$= 3.33$

$\therefore$ b is better!

3.10 $\dot{x} + 3x = u$ where $u(t)$ is an impulse

$\cancel{sX+}$ $x(t)$ $\mathcal{L}[x(t)] = X(s)$

$sX(s) + 3X(s) = 1$

$X(s) = \frac{1}{s+3} \Rightarrow x(t) = e^{-3t}$ ~~causal and stable~~

~~not sure how to get the other one...~~

causal & stable : $e^{-3t}u(t)$

anticausal & unstable: $-e^{-3t}u(-t)$

3.11 $S_x(s) = \frac{1 - s^2}{s^4 - 5s^2 + 4}$ $S_v(s) = 1$

a) optimal noncausal

$$G(\omega) = \frac{S_x(\omega)}{S_x(\omega) + S_v(\omega)} = \frac{\frac{1-s^2}{s^4 - 5s^2 + 4}}{\frac{1 - s^2 + s^4 - 5s^2 + 4}{s^4 - 5s^2 + 4}} = \frac{1 - s^2}{s^4 - 6s^2 + 5}$$

$$= \frac{1 - s^2}{(s^2 - 5)(s^2 - 1)} = \boxed{\frac{1}{5 - s^2}} \text{ or } \frac{1}{5 - (j\omega)^2} = \frac{1}{5 + \omega^2}$$

i think...

b) optimal causal $S_{xv}(\omega) = \frac{s^4 - 6s^2 + 5}{s^4 - 5s^2 + 4} = \frac{s^2 - 5}{s^2 - 4} = \frac{(j\omega)^2 - 5}{(j\omega)^2 - 4} = \frac{\omega^2 + 5}{\omega^2 + 4}$

$$= \frac{(s+\sqrt{5})(s-\sqrt{5})}{(s+2)(s-2)}$$

$$G(s) = \frac{s+2}{s+\sqrt{5}}\left[\text{causal of } \frac{\frac{-1+s^2}{(s^2-4)(s^2-1)}}{\frac{(s-\sqrt{5})}{(s-2)}}\right]$$

$$= \frac{s+2}{s+\sqrt{5}}\left[\frac{s-2}{s-\sqrt{5}}\frac{1}{(2-s)(2+s)}\right]$$

$$\boxed{= \frac{1}{s+\sqrt{5}}}$$

3.12 $G(s) = \frac{1}{s-3}$

similar to 3.10

$\Rightarrow e^{-3t}u(t)$

$\Rightarrow -e^{-3t}u(-t)$