4.1. Prove $\frac{d}{dt}\left(E[x]\right) = E\left[\frac{dx}{dt}\right]$

derivative and expectation are both linear
therefore interchangeable

4.2 dynamic scalar system $x_{k+1} = f x_k + w_k$

w_k has variance q. If $var(x_k) = \sigma^2$, show $f^2 = \frac{\sigma^2 - q}{\sigma^2}$

$$x_{k+1} = f x_k + w_k$$

$$\bar{x}_{k+1} = f \bar{x}_k$$

$$E\left[(x_{k+1} - \bar{x}_{k+1})^2\right] = E\left[\left(f(x_k - \bar{x}_k) + w_k\right)^2\right]$$

$$= E\left[f^2 (x_k - \bar{x}_k)^2\right] + 2E\left[f(x_k - \bar{x}_k) w_k\right] + E\left[w_k^2\right]$$

$$\sigma^2 = f^2\sigma^2 + 0 + q$$

$$\boxed{\Rightarrow f^2 = \frac{\sigma^2 - q}{\sigma^2}}$$

4.3 $x_k = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_{k-1}$, $w_k \sim (0,1)$

a) $\bar{x}_k = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \bar{x}_{k-1} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}^k \bar{x}_0$

$$= \begin{bmatrix} 1 & 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \left(\frac{1}{2}\right)^{k-1} \\ 0 & \frac{1}{2^k} \end{bmatrix} \bar{x}_0 =$$

$$\boxed{\lim_{k \to \infty} \bar{x}_k = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \bar{x}_0}$$

$$\bar{x} = \begin{bmatrix} \text{anything} \\ 0 \end{bmatrix}$$

b) $\cancel{P_k} = P_{steadystate} = \sum_{i=0}^{\infty} F^i Q (F^T)^i$

$Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & d \\ b & d \end{bmatrix} = \begin{bmatrix} b^2 & bd \\ bd & d^2 \end{bmatrix}$

$F^k = \begin{bmatrix} 1 & 2(1 - \frac{1}{2}^k) \\ 0 & \frac{1}{2}^k \end{bmatrix}$; $P_{steady\ state} = \sum_{i=0}^{\infty} \begin{bmatrix} 4(1 - \frac{1}{2}^i)^2 & 2(1 - \frac{1}{2}^k)\frac{1}{2}^k \\ 2(1 - \frac{1}{2}^k)\frac{1}{2}^k & \frac{1}{2}^{2k} \end{bmatrix}$

$$\boxed{\lim_{k \to \infty} P_{steadystate} = \begin{bmatrix} \infty & 1.33 \\ 1.33 & 1.33 \end{bmatrix}} = \begin{bmatrix} \sum_{i=0}^{\infty} 4 - \sum_{i=0}^{\infty} 4 \frac{1}{4}^i & \sum 2\frac{1}{2}^k - \sum 2\frac{1}{4}^k \\ \sum 2\frac{1}{2}^k - \sum 2\left(\frac{1}{4}\right)^k & \sum \frac{1}{4}^k \end{bmatrix}$$

4.4 skip

4.5 $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} w$

$$Q_c = L_k I L_k^T = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

a) $Q_{k-1} = \int_{t_{k-1}}^{t_k} e^{A(t_k - \tau)} Q_c(\tau) e^{A^T(t_k - \tau)} d\tau$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} e^{-(t_k-\tau)} & 0 \\ 0 & e^{-2(t_k-\tau)} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} e^{-(t_k-\tau)} & 0 \\ 0 & e^{-2(t_k-\tau)} \end{bmatrix} d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} 2e^{-2(t_k-\tau)} & 3e^{-3(t_k-\tau)} \\ 3e^{-3(t_k-\tau)} & 5e^{-4(t_k-\tau)} \end{bmatrix} d\tau = \left.\begin{bmatrix} e^{-2(t_k-\tau)} & e^{-3(t_k-\tau)} \\ e^{-3(t_k-\tau)} & \frac{5}{4}e^{-4(t_k-\tau)} \end{bmatrix}\right|_{t_{k-1}}^{t_k}$$

$$\boxed{Q_{k-1} = \begin{bmatrix} 1 - e^{-2(t_k - t_{k-1})} & 1 - e^{-3(t_k - t_{k-1})} \\ 1 - e^{-3(t_k - t_{k-1})} & \frac{5}{4}\left(1 - e^{-4(t_k - t_{k-1})}\right) \end{bmatrix}}$$

b) $Q_{k-1} \approx Q_c(t_k)\Delta t \approx \boxed{\begin{bmatrix} 2\Delta t & 3\Delta t \\ 3\Delta t & 5\Delta t \end{bmatrix}}$

where $\Delta t = t_k - t_{k-1}$

c) For $\Delta t = 0.1$

$Q_{k-1\,a} = \begin{bmatrix} 0.187 & 0.26 \\ 0.26 & 0.41 \end{bmatrix}$ $Q_{k-1\,b} = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}$

For $\Delta t = 0.01$

$Q_{k-1\,a} = \begin{bmatrix} 0.0198 & 0.0296 \\ 0.0296 & 0.049 \end{bmatrix}$ $Q_{k-1\,b} = \begin{bmatrix} 0.02 & 0.03 \\ 0.03 & 0.05 \end{bmatrix}$

4.6 $P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$

$$= \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} P_{k-1} \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} + \begin{bmatrix} 2T & 3T \\ 3T & 5T \end{bmatrix}$$

Note: $\begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} ae^{-2T} & be^{-3T} \\ ce^{-3T} & de^{-4T} \end{bmatrix}$

$$P_k = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix}^k P_0 \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix}^k +$$

$$\underbrace{\begin{bmatrix} 2T & 3T \\ 3T & 5T \end{bmatrix} *}_{\text{scalar mult.}} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} e^{-2T} & e^{-3T} \\ e^{-3T} & e^{-4T} \end{bmatrix} + \dots + \begin{bmatrix} e^{-2(k-1)T} & e^{-3(k-1)T} \\ e^{-3(k-1)T} & e^{-4(k-1)T} \end{bmatrix} \right)$$

$$\lim_{k \to \infty} P_k = 0 * P_0 * 0 + \begin{bmatrix} 2T * \frac{1}{1-e^{-2T}} & 3T \frac{1}{1-e^{-3T}} \\ 3T * \frac{1}{1-e^{-3T}} & 5T * \frac{1}{1-e^{-4T}} \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \frac{2T}{1-e^{-2T}} & \frac{3T}{1-e^{-3T}} \\ \frac{3T}{1-e^{-3T}} & \frac{5T}{1-e^{-3T}} \end{bmatrix} \approx \begin{bmatrix} 1 & 1 \\ 1 & 5/4 \end{bmatrix} \text{ for small } T}$$

4.7 $P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$

$$= \begin{bmatrix} e^{-\Delta t} & 0 \\ 0 & e^{-2\Delta t} \end{bmatrix} P_{k-1} \begin{bmatrix} e^{-\Delta t} & 0 \\ 0 & e^{-2\Delta t} \end{bmatrix} + \begin{bmatrix} 1-e^{-2\Delta t} & 1-e^{-3\Delta t} \\ 1-e^{-3\Delta t} & \frac{5}{4}(1-e^{-4\Delta t}) \end{bmatrix}$$

following the same idea/ principle in 4.6

$$= \begin{bmatrix} e^{-k\Delta t} & 0 \\ 0 & e^{-2k\Delta t} \end{bmatrix} P_0 \begin{bmatrix} e^{-k\Delta t} & 0 \\ 0 & e^{-2k\Delta t} \end{bmatrix} +$$

$$\begin{bmatrix} \cancel{(1-e^{-2\Delta t})} \frac{1-e^{-2k\Delta t}}{\cancel{1-e^{-2\Delta t}}} & \cancel{(1-e^{-3\Delta t})} \frac{1-e^{-3k\Delta t}}{\cancel{1-e^{-3\Delta t}}} \\ \cancel{(1-e^{-3\Delta t})} \frac{1-e^{-3k\Delta t}}{\cancel{1-e^{-3\Delta t}}} & \frac{5}{4}\cancel{(1-e^{-4\Delta t})} \frac{1-e^{-4k\Delta t}}{\cancel{1-e^{-4\Delta t}}} \end{bmatrix}$$

$$\boxed{P_k = \begin{bmatrix} P_{0_{11}} e^{-2\Delta T} & P_{0_{21}} e^{-3\Delta T} \\ P_{0_{12}} e^{-3\Delta T} & P_{0_{22}} e^{-4\Delta T} \end{bmatrix} + \begin{bmatrix} 1-e^{-2k\Delta t} & 1-e^{-3k\Delta t} \\ 1-e^{-3k\Delta t} & \frac{5}{4}(1-e^{-4k\Delta t}) \end{bmatrix}}$$

$\dot{P} = AP + PA^T + Q_c$

P4 $= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$\dot{P}_{11} = -2P_{11} + 2$ $\quad \dot{P}_{22} = -4P_{22} + 5$

$\dot{P}_{12} = -3P_{12} + 3$

$P_{11} = 1 - e^{-2t}$ $\quad P_{22} = \frac{5}{4}(1-e^{-4t})$

$P_{12} = 1 - e^{-3t}$

4.8 $x_{k+1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} x_k + w_k$ $\quad Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$P = \sum_{i=0}^{\infty} F^i Q (F^T)^i = \sum_{i=0}^{\infty} \begin{bmatrix} \frac{1}{2}^i & 0^i \\ 0^i & \frac{1}{2}^i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}^i & 0^i \\ 0^i & \frac{1}{2}^i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix} (1/4)^i & (1/4)^i \\ 0 & (1/4)^i \end{bmatrix} = \begin{bmatrix} \frac{1}{1-1/4} & \frac{1}{1-1/4} \\ 0 & \frac{1}{1-1/4} \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 4/3 & 4/3 \\ 0 & 4/3 \end{bmatrix}}$$

4.9

$F = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

4.10 $\int_{-\infty}^{\infty} f(t)\,\delta(t-\alpha)\,dt = \int_{-\infty}^{\infty} f(\alpha)\,\delta(t-\alpha)\,dt$

$= \underline{f(\alpha)}$ //