5.1 a)
$$x_k = \tfrac{1}{2}x_{k-1} + w_k$$
$$y_k = \underbrace{1}_{H_k}\, x_k + v_k$$

b)
$$x_k^+ = (I - K_kH_k)(F_{k-1}x_{k-1}^+ + G_{k-1}u_{k-1}) + K_k y_k$$
$$K_k = P_k^+H_k^TR_k^{-1} = \frac{P_k^+}{R}$$
$$x_k^+ = \left(1 - \frac{P_k^+}{R}\right)\left(\tfrac{1}{2}x_{k-1}^+\right) + \frac{P_k^+}{R}y_k$$

$$\cancel{c)\ P_k^+ = (I - K_kH_k)(F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1})}$$
$$\cancel{= \left(1 - \frac{P_k^+}{R}\right)\left(\tfrac{1}{4}P\right)}$$

$$K_k = P_k^-H_k^T(H_kP_k^-H_k^T + R_k)^{-1}$$
$$= \frac{P_k^-}{P_k^- + R}$$
$$P_k^- = F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1} = \tfrac{1}{4}P_{k-1}^+ + Q$$
$$K_k = \frac{\frac{1}{4}P_{k-1}^+ + Q}{\frac{1}{4}P_{k-1}^+ + Q + R}$$
$$x_k^+ = \left(\frac{R}{\frac{1}{4}P_{k-1}^+ + Q + R}\right)\left(\tfrac{1}{2}x_{k-1}^+\right) + \left(\frac{\frac{1}{4}P_{k-1}^+ + Q}{\frac{1}{4}P_{k-1}^+ + Q + R}\right)y_k$$

c)
$$P_k^+ = (I - K_kH_k)(F_{k-1}P_{k-1}^+F_{k-1} + Q_{k-1})$$
$$= \left(\frac{R}{\frac{1}{4}P_{k-1}^+ + Q + R}\right)\left(\tfrac{1}{4}P_{k-1}^+ + Q\right)$$

d)
$$K_k = \frac{\frac{1}{4}P_\infty^+ + Q}{\frac{1}{4}P_\infty^+ + Q + R}$$

If $Q = R$
$$K_k = \frac{\frac{1}{4}P_\infty^+ + R}{\frac{1}{4}P_\infty^+ + 2R}$$

If $Q = 2R$
$$K_k = \frac{\frac{1}{4}P_\infty^+ + 2R}{\frac{1}{4}P_\infty^+ + 3R}$$

Assuming $R > 1$, $K_k @ Q = 2R > K_k @ Q = R$

Intuitive Explanation: we will trust our measurement y more if our measurement noise R is much less than our model noise Q.

$c_{cont.})$
$$P_\infty^+ = \frac{R(P_\infty^+ + 4Q)}{P_\infty^+ + 4(Q+R)} \Rightarrow P_\infty^{+2} + (4Q+3R)P_\infty^+ - 4QR = 0$$
$$P_\infty^+ = \frac{-(4Q+3R) \pm \sqrt{(4Q+3R)^2 + 16QR}}{2}$$

$d_{cont.})$ If $Q = R$, $P_\infty^+ = 0.5311$ or $\cancel{-7.5311}$ $\quad K_k = \frac{P_\infty^+}{R} = 0.533$

If $Q = 2R$, $P_\infty^+ = 0.6847$ or $\cancel{-5.8423}$ $\quad K_k = 0.6847$

5.2 b)
$$P_k^+ = (I - K_kH_k)P_k^-(I - K_kH_k)^T + K_kR_kK_k^T$$
$$K_k = P_k^+H_k^TR_k^{-1} = n \times m$$
x n state

y = m states
$$= P_k^-H_k^T(H_kP_k^-H_k^T + R_k)^{-1}$$

$$P_k^+ = (P_k^- - K_kH_kP_k^-)(I - H_k^TK_k^T)$$
$$+ K_kR_kK_k^T$$
$$= P_k^- - P_k^-H_k^TK_k^T - K_kH_kP_k^- + K_kH_kP_k^-H_k^TK_k^T$$
$$+ K_kR_kK_k^T$$

$$\frac{\partial Tr(P_k^+)}{\partial K_k} = 0 - \overbrace{2P_k^-H_k^T}^{\text{not sure}} + 2K_kH_kP_k^-H_k^T$$
$$+ 2K_kR_k$$
$$= -2P_k^-H_k^T + 2K_k(H_kP_k^-H_k^T + R_k)$$
$$= -2P_k^-H_k^T + 2P_k^-H_k^T\frac{\cancel{(H_kP_k^-H_k^T + R_k)}}{\cancel{(H_kP_k^-H_k^T + R_k)}}$$
$$= 0$$

[for a scalar system!!!]

a)
$$P_k^+ = (I - K_kH_k)P_k^- = P_k^- - K_kH_kP_k^-$$
$$\frac{\partial Tr(P_k^+)}{\partial K_k} = -H_k^TP_k^-$$

c) The Joseph form has a 0 derivative (it will not change wrt to K_k) while the 3rd form changes.

5.3
$$x_k = Fx_{k-1} + Gu_{k-1} + w_{k-1}$$
$$\hat{x}_k = F\hat{x}_{k-1} + Gu_{k-1}$$

Prove $E[\hat{x}_1^+(\tilde{x}_1^+)^T] = 0$

Induction base case

Note $\hat{x}_0^+ = E[x_0]$ constant

$E[\hat{x}_0^+(\tilde{x}_0^+)^T] = 0$

$$\hat{x}_1^+ = F\hat{x}_0 + Gu_{k-1}$$

since $\hat{x}_0$ is constant, so does G and u_{k-1}, then $\hat{x}_1^+$ is also constant

$\tilde{x}_1^+ = \cancel{F x_0 + Gu_{k-1}}$ which is also zero mean

$F\tilde{x}_0^+ + w_{k-1}$

Hence $E[\hat{x}_1^+(\tilde{x}_1^+)^T] = 0$

Induction case can be proven similarly.

[proof is a bit lacking]

5.4 a) $x_p(k+1) = x_p(k) - k_2 x_p(k) + k_1 x_g(k) + w_p(k)$

$x_g(k+1) = -k_3 x_p(k) + x_g(k) + u(k) + w_g(k)$

$x = \begin{bmatrix} x_p \\ x_g \end{bmatrix} \quad x_k = \begin{bmatrix} 1-k_2 & k_1 \\ -k_3 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k-1} + w_{k-1}$

$y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + v_k$

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$x_k = \begin{bmatrix} 1/2 & 1 \\ -1/2 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k-1} + w_{k-1}$

b) Since at initial time we have perfect count, $P_0 = 0$

$P_1^- = 0 + Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P_2^- = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$

$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad K_2 = \begin{bmatrix} 1 & 0 \\ 0.1429 & 0.5714 \end{bmatrix}$

$P_1^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} \quad P_2^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0.5714 \end{bmatrix}$

c) ~~$\frac{0}{0} = 1$ after some point, the guppy will just die out then the piranhas will die out as well~~

$x_p = \frac{1}{2} x_p + x_g$

$x_g = -\frac{1}{2} x_p + x_g + u$

$x_p = 2 x_g \quad \Rightarrow \quad \frac{x_p}{x_g} = \boxed{2}$

$x_p = 2u$

5.5 $y_k = z_k + z_{k-1}$

a) $x_k = \begin{bmatrix} z_{k-1} \\ z_k \end{bmatrix} \quad x_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} z_{k+1}$

$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + R \quad x_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k-1} + w_k$

$R = 0 \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

b) $P_0^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(i) Base Case $P_1^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$K_1^+ = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$P_1^+ = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(ii) Induction case: assume $P_k^+ = \frac{1}{k+1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is true, show P_{k+1}^+ is also true.

$P_{k+1}^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} P_k^+ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/(k+1) & 0 \\ 0 & 1 \end{bmatrix}$

$K_{k+1} = P_{k+1}^- H_k^T (H_k P_{k+1}^- H_k^T + R_k)^{-1}$

$= \begin{bmatrix} \frac{1}{k+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{k+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \right)^{-1}$

$= \begin{bmatrix} \frac{1}{k+1} \\ 1 \end{bmatrix} \left(\frac{k+2}{k+1} \right)^{-1} = \begin{bmatrix} \frac{1}{k+2} \\ \frac{k+1}{k+2} \end{bmatrix}$

$P_{k+1}^+ = (I - K_k H_k) P_{k+1}^-$

$= \begin{bmatrix} \frac{k+1}{k+2} & \frac{-1}{k+2} \\ \frac{-(k+1)}{k+2} & \frac{1}{k+2} \end{bmatrix} \begin{bmatrix} \frac{1}{k+1} & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{k+2} & -\frac{1}{k+2} \\ -\frac{1}{k+2} & \frac{1}{k+2} \end{bmatrix}$

$= \frac{1}{k+2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

c) $E\left[\| x_k - \hat{x}_k^+ \|_2^2 \right] = E\left[Tr\{ (x_k - x_k^+)(x_k - x_k^+)^T \} \right] = Tr(P_k^+)$

$= \frac{2}{k+1}$

5.6 $\begin{bmatrix} I & 0 \\ -P_k^- H_k^T S_k^{-1} & I \end{bmatrix} \begin{bmatrix} S_k & H_k P_k^- \\ P_k^- H_k^T & P_k^- \end{bmatrix} = \begin{bmatrix} S_k & H_k P_k^- \\ 0 & P_k^+ \end{bmatrix}$

$|I - 0| \, |\ldots| = |S_k| \, |P_k^+|$

$|P_k^-| \, |S_k - H_k P_k^- H_k^T| = |S_k| \, |P_k^+|$

$\Rightarrow |P_k^+| = \frac{|P_k^-| \, |R_k|}{|S_k|}$

5.7 $\Sigma_K = E\left[(x_K - \bar{x}_K)(x_K - \bar{x}_K)^T\right]$ state covariance

$$\Sigma_{K+1} = F_K \Sigma_K F_K^T + Q_K$$

$$P_K = E\left[(x - \hat{x}^+)(x - \hat{x}^+)\right]$$ estimation error covariance

$$P_{K+1}^- = F_K P_K^- F_K^T + Q_K - F_K P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} H_K P_K^- F_K^T$$

Show $\Sigma_K - P_K^- \geq 0$ for all K

$$\Sigma_K - P_K^- = F_K P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} H_K P_K^- F_K^T$$

Given that P_K^- and R_K are positive definite, $H_K P_K^- H_K^T$ is also positive definite, and so is $\Sigma_K - P_K^-$. $\therefore \Sigma_K - P_K^- \geq 0$

Intuitive Explanation: The state covariance is always greater than the ~~state~~ estimate covariance, ~~[illegible]~~ where covariance is a measure of "assurance".

5.8
$$x_K = \tfrac{1}{2} x_{K-1} + w_{K-1} \qquad w_K \sim N(0, Q)$$
$$y_K = x_K + v_K \qquad v_K \sim N(0, R)$$

$$\begin{bmatrix} A_{K+1} \\ B_{K+1} \end{bmatrix} = \begin{bmatrix} (F_K + Q_K F_K^{-T} H_K^T R_K^{-1} H_K) & Q_K F_K^{-T} \\ F_K^{-T} H_K^T R_K^{-1} H_K & F_K^{-T} \end{bmatrix} \begin{bmatrix} A_K \\ B_K \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{1}{2} + Q(2)(1)R^{-1}(1) & 2Q \\ 2R^{-1} & 2 \end{bmatrix}$$

Given $Q=1$, $R=5$, $P_0=0$

$$\begin{bmatrix} A_{K+1} \\ B_{K+1} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} + 2QR^{-1} & 2Q \\ 2R^{-1} & 2 \end{bmatrix} \begin{bmatrix} A_K \\ B_K \end{bmatrix} \qquad \Phi = \begin{bmatrix} \cancel{1.9/5}\ 9/10 & 2 \\ \cancel{1/5}\ 2/5 & 2 \end{bmatrix}$$

$$P_K^- = A_K B_K^{-1} \quad \text{where} \quad \begin{bmatrix} A_K \\ B_K \end{bmatrix} = \Phi^K \begin{bmatrix} P_0 \\ I \end{bmatrix}$$

$$\Phi = V * D * V^{-1} \Rightarrow V = \begin{bmatrix} -0.9701 & -0.7809 \\ 0.2425 & -0.6247 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.4 & 0 \\ 0 & 2.5 \end{bmatrix}$$

$$\Phi^K = \begin{bmatrix} -0.9701 & -0.7809 \\ 0.2425 & -0.6247 \end{bmatrix} \begin{bmatrix} 0.4^K & 0 \\ 0 & 2.5^K \end{bmatrix} \begin{bmatrix} -0.7854 & 0.9817 \\ -0.3049 & -1.2196 \end{bmatrix}$$

$$= \begin{bmatrix} -0.9701(0.4^K) & -0.7809(2.5^K) \\ 0.2425(0.4)^K & -0.6247(2.5)^K \end{bmatrix} \begin{bmatrix} -0.7854 & 0.9817 \\ -0.3049 & -1.2196 \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup & -0.9524(0.4)^K + 0.9524(2.5^K) \\ \sqcup & 0.2381(0.4)^K + 0.7619(2.5)^K \end{bmatrix}$$

$$\begin{bmatrix} A_K \\ B_K \end{bmatrix} = \Phi^K \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9524(0.4)^K + 0.9524(2.5)^K \\ 0.2381(0.4)^K + 0.7619(2.5)^K \end{bmatrix}$$

$$P_K^- = \frac{-0.9524(0.4)^K + 0.9524(2.5)^K}{0.2381(0.4)^K + 0.7619(2.5)^K}$$

b) $$\lim_{K\to\infty} P_K^- = \frac{0.9524}{0.7619} = \underline{1.25}$$

5.9 a) $x_{K+1} = x_K$

$y_K = x_K + v_K$ where $v_K \sim (0, R)$

$\Rightarrow F = H = 1 \quad Q_K = 0$

$$P_{K+1}^- = F_K P_K^- F_K^T + Q_K - F_K P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} H_K P_K^- F_K^T$$
$$= P_K^- - P_K^- (P_K^- + R)^{-1} P_K^- = P_K^- \left(1 - \frac{P_K^-}{P_K^- + R}\right)$$

$$P_{K+1}^- = \frac{P_K^- R}{P_K^- + R} \qquad P_0^+ = 1 \text{ (assuming } x_0 = 0)$$

$$P_K^- = \frac{P_0^- R}{R + K P_0^-}$$ ← obtained by trying P_1, P_2, P_3 and then figuring out the pattern

$$\boxed{P_K^- = \frac{R}{R+K}} \qquad \lim_{K\to\infty} P_K^- = \frac{R}{R+K} = \underline{0}$$

b) $x_{K+1} = x_K + w_K \qquad w_K \sim (0, Q)$

$${}^b P_{K+1}^- = P_K^- + Q - P_K^- (P_K^- + R)^{-1} P_K^-$$
$$= \frac{P_K^-(P_K^- + R) + Q(P_K^- + R) - P_K^-}{P_K^- + R}$$

$${}^b P_{K+1}^- = \frac{P_K^- R + Q(P_K^- + R)}{P_K^- + R}$$

let the P_{K+1}^- of a) be ${}^a P_{K+1}^-$

$${}^b P_{K+1}^- - {}^a P_{K+1}^- = Q$$ ← let us call this ΔP_{K+1}^-

$$\lim_{K\to\infty} \Delta P_K^- = \underline{Q}$$

$$\hat{x}_{K+1}^- = \hat{x}_K^+ = \hat{x}_K^- + K_K(y_K - H_K \hat{x}_K^-)$$

$$K_K = P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} = \frac{P_K^-}{P_K^- + R}$$

note that $P_K^- = \frac{R}{R+K}$

$$= \frac{\frac{R}{R+K}}{\frac{R}{R+K} + \frac{R(R+K)}{R+K}} = \frac{1}{R+K+1}$$

$$\hat{x}_{K+1}^- = \hat{x}_K^- + \frac{1}{R+K+1}\left(x_K + v_K - \hat{x}_K^-\right)$$

$$= \frac{R+K}{R+K+1} \hat{x}_K^- + \frac{x_K}{R+K+1} + \frac{v_K}{R+K+1}$$

$$\epsilon_{K+1} = x_{K+1} - \hat{x}_{K+1}^- = x_K + w_K - \hat{x}_{K+1}$$

$$= \frac{R+K}{R+K+1} \epsilon_K + w_K - \frac{v_K}{R+K+1}$$

$$E[\epsilon_{K+1}^2] = \left(\frac{R+K}{R+K+1}\right)^2 E[\epsilon_K^2] + Q - \frac{R}{(R+K+1)^2}$$

$$\lim_{K\to\infty} E[\epsilon_K^2] = \infty$$ because of "Q".

5.10 The gain [illegible] assumed [illegible] the same for both...

$$P_k^- = F_{k-1}(I - K_{k-1}H_{k-1})P_{k-1}^-(I - K_{k-1}H_{k-1})^T + F_{k-1}K_{k-1}R_k K_{k-1}^T F_{k-1}^T + Q_{k-1}$$

$$\Delta_k = \Sigma_k - P_k^- = F_{k-1}(I - K_{k-1}H_{k-1})\Delta_k(I - K_{k-1}H_{k-1})^T + F_{k-1}K_{k-1}\Delta R K_{k-1}^T F_{k-1}^T$$

If ΔR is positive definite, then Δ_k is positive definite.

5.11
$$\begin{bmatrix} p_k \\ f_k \end{bmatrix} = \begin{bmatrix} 0.5 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_{k-1} \\ f_{k-1} \end{bmatrix} + w_{k-1}$$

$$w_{k-1} \sim \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}\right)$$

$$y_k = p_k + v_k$$

$$v_k \sim (0, 10)$$

5.12 skip