6.1

a) Sequential | Batch
- no inverse multiplication
- no R limitations
- diag R / const R requirement

b) Information F | Standard KF
- $r \gg n$
- more precise if full knowledge
- more precise if ~~...~~ no knowledge
- $n \gg r$

c) Square root | Standard KF
- more precision (2x)
- less computation complexity

6.2 $R = S\hat{R}S^{T}$

$$S = \cancel{\begin{bmatrix} 0.9701 & -0.7809 \\ 0.2425 & -0.6247 \end{bmatrix}} \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\bar{H}_K = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$\bar{y}_K = \bar{H}_K x_K + S^{-1} v_K$$

6.3 Eq 6.28 seems ~~computations~~ simpler, Fewer operations, (2 matrix inv.) but each operation requires a lot of computation. Easier to implement (in a way).

*Add: guarantees $\mathcal{I}_K$ will be symm positive definite if $\mathcal{I}^+_{K-1}$ is symm positive definite.

Eq. 6.30 seems more complicated (longer eq.). However if Q_K is constant, solving for Q^{-1}_{K-1} shouldn't be expensive, hence we can do some preprocessing and make the implementation do less computation (only 1 matrix inversion)

6.4 a) $x_K = \frac{1}{2}x_{K-1} + w_K$

$$y_K = \begin{bmatrix} 1 \\ \phi \end{bmatrix} x_K + v_K \qquad v_K \sim \left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$\hat{x}^+_0 = 0$

$\mathcal{I}^+_0 = P_0^{-1} = 1$

$\mathcal{I}^-_1 = Q_0^{-1} - Q_0^{-1}(\frac{1}{2})(\mathcal{I}^+_0 + \frac{1}{2}Q_0^{-1}\frac{1}{2})^{-1}\frac{1}{2}Q_0^{-1} = 0.8$

$\mathcal{I}^+_1 = \mathcal{I}^-_1 + H_K^T R_K^{-1} H = 2.8$

$\mathcal{I}^-_2 = Q_1^{-1} - Q_1^{-1}(\frac{1}{2})(\mathcal{I}^+_1 + \frac{1}{4}Q_1^{-1})^{-1}\frac{1}{2}Q_1^{-1} = 0.7467$

$\mathcal{I}^+_2 = \mathcal{I}^-_2 + H_K^T R_K^{-1} H = 2.7467$

b) let $y_0 = N(0, \sigma^2)$, $y_1 = N(0, \sigma^2)$

mean $E\left[\frac{y_1 + y_2}{2}\right] = 0$

var $E\left[\left(\frac{y_1+y_2}{2}\right)^2\right] = \frac{1}{4}\left[E[y_1^2] + E[y_2^2]\right] = \frac{\sigma^2}{2}$

$x_0^+ = 0$

$P_0^+ = 1$

$P_1^- = \frac{1}{2}(\phi)\frac{1}{2} + Q_0 = 1.25$

$K_1 = 1.25 \cdot 1 (1.25 + \frac{1}{2})^{-1} = \cancel{0.667}\ 0.7143$

$P_1^+ = 0.3571 \qquad (I - K_1 * 1) P_1^-$

$P_2^- = \frac{1}{4}P_1^+ + Q_1 = 1.3393$

$K_2 = P_2^- * (P_2^- + \frac{1}{2})^{-1} = \cancel{1.8830}\ 0.7282$

$P_2^+ = (I - K_2) P_2^- = 0.3641$

Note: $P_1^-, P_1^+, P_2^-, P_2^+$ are indeed the inverse of $\mathcal{I}$s

6.5 Let $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

$$\sigma^2(D) = \lambda(D^T D) = \lambda\left(\begin{bmatrix} d_1^2 & & \\ & \ddots & \\ & & d_n^2 \end{bmatrix}\right)$$

Note that the eigenvalues of D^TD are the diagonals itself.

$$\sigma(D) = \lambda\left(\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}\right)$$

6.6 If $x^TAx \geq 0$ for all $n \times 1$ x, A is positive ~~sem~~ definite.

Base case: I is positive definite because $x^Tx \geq 0$ for all $n \times 1$ x.

Notice x^TSS^Tx, S is $n \times n$ so x^TS is also $1 \times n$ or $x' = S^Tx$ is $n \times 1$.

Since for $A = I$, all $x^Tx \geq 0$ where x is all $n \times 1$ matrix, it means $x' = S^Tx$ is included ~~to~~ in all x^Tx. ~~hence~~

$\therefore$ since $x^Tx \geq 0 \Rightarrow x^TSS^Tx \geq 0$

$\therefore SS^T$ is positive definite

$$(SS^T)^T = S^{T^T}S^T = SS^T$$

$\therefore SS^T$ is symmetric

~~6.7~~ $S = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0.707 & 0.707 \\ 2.053 & 2.187 \end{bmatrix}$

$a=1\ b=0$ $\qquad$ $a=b=\sqrt{0.5}$

~~∴ the sol'n is not unique.~~

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

~~6.8~~ $\cancel{SS^T = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}}$

$\cancel{i=1}$

$\cancel{S_{11} = \sqrt{P_{11}} = \sqrt{5}}$

6.7 Find an upper triangle matrix S

$$SS^T = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

Let $S = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $SS^T = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a^2+b^2 & bc \\ bc & c^2 \end{bmatrix}$

$c = \pm 3 \quad b = \pm 1 \quad a = 0$

My sol'n is not unique because I have two.

$S = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix}$

6.8 $SS^T = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

$S = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ $\quad SS^T = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} = \begin{bmatrix} a^2+b^2+c^2 & bd+ce & cf \\ bd+ce & d^2+e^2 & ef \\ cf & ef & f^2 \end{bmatrix}$

$f = \pm 1 \quad e = \mp 1 \quad c = \mp 2 \;\Big|\; d = \pm 1,\ b = 0 \;\Big|\; a = \pm 1$

There are $2^3 = 8$ possible sol'n.

$$\begin{bmatrix} \pm 1 & 0 & -2f \\ 0 & \pm 1 & -f \\ 0 & 0 & \pm 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

6.9 (eq 6.70) $I - a\phi\phi^T = (I - a\gamma\phi\phi^T)^2$

where $\gamma = \dfrac{1}{1 \pm \sqrt{aR_{ik}}}$ $\quad a = \dfrac{1}{\phi^T\phi + R_{ik}}$

$\phi = S^{+T}_{i-1,k} H^T_{ik}$ $\qquad$ S is $n \times n$, H_{ik} is $1 \times n$, ϕ is $n \times 1$

~~n×n r×1 n×1 n×1 1×n r×n n×1~~

~~K = n×r r×n~~

$$I - a\phi\phi^T = I - 2a\gamma\phi\phi^T + a^2\gamma^2\phi\phi^T\phi\phi^T$$

$$0 = (1 - 2a\gamma)\cancel{\phi\phi^T} a + \cancel{a^2}\gamma^2\phi\phi^T\cancel{\phi\phi^T}$$

~~Trace~~ For the right side eq. to be zero, the sum of the trace of its components must be also zero, so that the resulting matrix's eigenvalues will be all zero.

$$0 = (1 - 2a\gamma) + a\phi^T\phi\,\gamma^2$$

$$\gamma = \frac{2 \pm \sqrt{4 - 4a\phi^T\phi}}{2a\phi^T\phi}$$

$$= \frac{1 \pm \sqrt{1 - a\phi^T\phi}}{a\phi^T\phi} = \frac{1 \pm \sqrt{aR_{ik}}}{1 - aR_{ik}} = \boxed{\frac{1}{1 \pm \sqrt{aR_{ik}}}}$$

6.10

$$\begin{bmatrix} u_1 & \cdots & u_{n+r} \end{bmatrix} = \bar{T}\begin{bmatrix} A_1 & \cdots & A_{n+r} \end{bmatrix}$$

$u_1 = \bar{T}A_1$ $\quad$ note $u_1 = \begin{bmatrix} u_{11} \\ 0 \\ 0 \\ \vdots \end{bmatrix}$

$u_1^T u_1 = (\bar{T}A_1)^T(\bar{T}A_1)$

$u_{11}^2 = A_1^T \underbrace{\bar{T}^T\bar{T}}_{I} A_1$

$|u_{11}| = \|A_1\|_2$

6.11 $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$

$k = 1$

$\sigma_1 = 1 * \sqrt{\sum_{i=1}^{4} (A_{i1})^2} = +3$

$\beta_1 = \frac{1}{12}$

$u^{(1)}_{1-4} = 4, 2, 0, 2$

$y^{(1)}_{1-2} = 1, 1$

$A^{(2)} = \begin{bmatrix} -3 & -3 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$T^{(1)} = \begin{bmatrix} -\frac{1}{3} & -2/3 & 0 & -2/3 \\ -2/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ -2/3 & -1/3 & 0 & 2/3 \end{bmatrix}$

$k = 2$

$\sigma_2 = +1 \sqrt{\sum_{i=2}^{4} (A^{(2)}_{i2})^2} = +1$

$\beta_1 = 1$

$u^{(2)} = [0, 1, 1\ 0]^T$

$y^{(2)} = [0, 1]^T$

$A^{(3)} = \begin{bmatrix} -3 & -3 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T = T^{(2)} T^{(1)} = \begin{bmatrix} -1/3 & -2/3 & 0 & -2/3 \\ 0 & 0 & -1 & 0 \\ 2/3 & -2/3 & 0 & 1/3 \\ -2/3 & -1/3 & 0 & 2/3 \end{bmatrix}$

6.12 Use Gram Schmidt method $A^{(1)} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$

For $k = 1$

$\sigma_1 = 3$

$w_1 = [3\ 3]$

$T_1 = [1/3\ 2/3\ 0\ 2/3]$

$A^{(2)} = \begin{bmatrix} x & 0 \\ x & 0 \\ x & 1 \\ x & 0 \end{bmatrix}$

for $k = 2$

$\sigma_2 = 1$

$w_2 = [0\ 1]$

$T_2 = [0\ 0\ 1\ 0]$

$T = \begin{bmatrix} 1/3 & 2/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & 1 \end{bmatrix}$

$T_3 = [0.9428\ -0.2357\ 0\ -0.2357]$

$T_4 = [0\ 0.707\ 0\ -0.707]$

$T = \begin{bmatrix} 1/3 & 2/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 0.9428 & -0.2357 & 0 & -0.2357 \\ 0 & 0.707 & 0 & -0.707 \end{bmatrix}$

6.13 $P = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = UDU^T$

$= \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \begin{bmatrix} u_{11} & 0 \\ u_{12} & u_{22} \end{bmatrix}$

$= \begin{bmatrix} u_{11} d_{11} & u_{12} d_{22} \\ 0 & u_{22} d_{22} \end{bmatrix} \begin{bmatrix} u_{11} & 0 \\ u_{12} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11}^2 d_{11} + u_{12}^2 d_{22} & u_{12} d_{22} u_{22} \\ u_{12} u_{22} d_{22} & u_{22}^2 d_{22} \end{bmatrix}$

~~by trial and error, we can get:~~

~~$u_{11} = 0$, $u_{12} = 1$, $u_{22} = 3$~~

~~$d_{11} =$ any (let it be 2) $d_{22} = 1$~~

$\cancel{P = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}}_{U^T}}$

* Note that we can assume u_{11} and $u_{22} = 1$

$P = \begin{bmatrix} d_{11} + u_{12}^2 d_{22} & u_{12} d_{22} \\ u_{12} d_{22} & d_{22} \end{bmatrix}$

$d_{22} = 9$, $u_{12} = 1/3$, $d_{11} = 0$

$P = \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix}$ //

6.14, 6.15 } skipping these for now as my thesis will unlikely need these. I need to spend more time w/ nonlinear KF.