7.1 $x_k = \frac{1}{2}x_{k-1} + w_{k-1}$ $\quad Q = Q_c = 1$

$y_k = x_k + v_k$

$v_k = \frac{1}{2}v_{k-1} + \delta_{k-1}$

a) Find P^+_∞

$$P_k^+ = (I - K_k H_k)(F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1}) \quad \text{one step KF}$$

$$P_k^- = F_k P_k^- F_k^T + Q_k - \frac{F_k P_k^- H_k^T H_k P_k^- F_k^T}{H_k P_k^- H_k^T + R_k} \quad \text{one step KF}$$

$$P_\infty^- = \frac{1}{4}P_\infty^- + 1 - \frac{\frac{1}{4}P_\infty^{-2}}{P_\infty^- + 1}$$

$$P_\infty^{-2} + \cancel{P_\infty^-} = \cancel{\tfrac{1}{4}P_\infty^{-2}} + \frac{1}{4}P_\infty^- + \cancel{P_\infty^-} + 1 - \cancel{\tfrac{1}{4}P_\infty^{-2}}$$

$$P_\infty^{-2} - \frac{1}{4}P_\infty^- - 1 = 0$$

$$P_\infty^- = \cancel{1.1328}\ \frac{1+\sqrt{65}}{8},\ \cancel{\frac{1-\sqrt{65}}{8}} \quad \text{negative}$$

$$K_\infty = \frac{-7+\sqrt{65}}{2}$$

$$\boxed{P_\infty^+ = \frac{-7+\sqrt{65}}{2}}$$

b) $E(e_k^2) = E\left((x_k - \hat{x}_k^+)^2\right)$

$$= E\left(\left(\tfrac{1}{2}x_{k-1} + w_{k-1} - (\hat{x}_k^- + K_k(y_k - H_k\hat{x}_k^-))\right)^2\right)$$

$$= E\left(\left(\tfrac{1}{2}x_{k-1} + w_{k-1} - \tfrac{1}{2}\hat{x}_{k-1}^+ - K_k\left(\tfrac{1}{2}x_{k-1} + w_{k-1} + v_k - \tfrac{1}{2}\hat{x}_{k-1}^+\right)\right)^2\right)$$

$$= E\left(\left(\tfrac{1}{2}\epsilon_{k-1} + w_{k-1} - K_k\left(\tfrac{1}{2}\epsilon_{k-1} + w_{k-1} + v_k\right)\right)^2\right)$$

$$= \Big[K_k^2 v_k^2 + \tfrac{1}{4}(K_k^2 - 2K_k + 1)\epsilon_{k-1}^2 + (K_k^2 - K_k)\epsilon_{k-1}v_k + (K_k^2 - 2K_k + 1)w_{k-1}^2 + \underbrace{\cancel{((K_k^2 - 2K_k + 1)\epsilon_{k-1} + 2(K_k^2 - K_k)v_k)w_k}}_{0 \text{ bec. of } w_k}\Big]$$

↳ obtained using sage

▶ ~~E(ε~~ Solve for $E(v_k^2) = E\left(\tfrac{1}{4}v_{k-1}^2 + \cancel{v_{k-1}\delta_{k-1}} + \delta_{k-1}^2\right)$

$$= \delta_{k-1}^2 + \tfrac{1}{4}\delta_{k-2}^2 + \ldots = 1 + \tfrac{1}{4} + \ldots$$

$$\lim_{k\to\infty} E(v_k^2) = \frac{1}{1 - 1/4} = 4/3$$

▶ Solve for $E(\epsilon_{k-1}v_k) = \cancel{E(x_{k-1}v_k)} - E(\hat{x}_{k-1}^+ v_k)$

$$= -\cancel{E(\hat{x}_{k-1}^- v_k)} - E\left(K_k(\cancel{x_{k-1}} + v_{k-1} - \cancel{\hat{x}_{k-1}})v_k\right)$$

$$= -E(K_k v_{k-1} v_k) = -\tfrac{1}{2}K_k E(v_{k-1}^2)$$

$$\lim_{k\to\infty} E(\epsilon_{k-1}v_k) = -\frac{2}{3}K_k$$

▶ Solve for $E(\epsilon_k^2)$

$$\epsilon_k^2 = \frac{4(2K_k^3 - 9K_k^2 + 6K_k - 3)}{3(K_k^2 - 2K_k - 3)}$$

from a) $K_\infty = \frac{-7+\sqrt{65}}{2}$

giving us $E(\epsilon_k^2) \approx 0.724$

c) Setup

$$F = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

dare(F', H', Q, 0) gives

$$P_\infty^- = \begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.1667 \end{bmatrix}$$

$$P_\infty^+ = \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix}$$

7.2 since noise free, $R = 0$

$$P_{k+1}^- = FP_k^-F^T + Q - FP_k^-H^T(HP_k^-H^T)^{-1}HP_k^-F^T$$

$$P_{k+1}^- = Q$$

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R)^{-1}$$

$$= QH_k^T(H_k Q H_k^T)^{-1}$$

$$P_k^+ = (I - K_k H_k)P_k^-$$

$$= \Big(I - \overbrace{\cancel{QH_k^T(H_kQH_k^T)^{-1}H_k}}^{I}\Big)P_k^-$$

$$P_k^+ = 0$$

Note: the sol'n manual did it differently

It solved for $P_k^+ = \ldots P_{k-1}^+ \ldots$

@ $R = 0$

Tried $P_k^+ = 0$ and obtained the eq.

$$(HQH^T)Q = QH^THQ$$

7.3 $x_K = x_{K-1} + w_{K-1}$

$y_K = x_K + v_K$

$Q = R = 1 \qquad M = 1$

a) Design KF ignoring $E(w_K v_{K+1}) = M = 1$, what is P_∞^+ ?

Using one step KF for P_{K+1}^-

$$P_{K+1}^- = F P_K^- F^T + Q - F P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} H_K P_K^- F_K^T$$

At steady state (dropping sub/super scripts)

$$P = \cancel{FPF^T+I}\ P + 1 - P(P+1)^{-1} P$$

giving us $P_\infty^- = \pm\frac{\sqrt{5}}{2} + \frac{1}{2}$

solving for K_∞ and P_∞^+ gives us

$P_\infty^+ = \frac{\pm\sqrt{5}-1}{2}$ and since P_∞^+ must be positive, $\boxed{P_\infty^+ \approx 0.6180}$

b) Suppose $e_K = x_K - \hat{x}_K^+$. Find $E(e_K^2)$

$$e_K = x_{K-1} + w_{K-1} - \hat{x}_K^- - K_K(y_K - x_K^-)$$

$$= x_{K-1} + w_{K-1} - \hat{x}_{K-1}^+ - K_K(x_{K-1} + w_{K-1} + v_K - x_{K-1}^+)$$

$$= e_{K-1} + w_{K-1} - K_K(e_{K-1} + w_{K-1} + v_K)$$

Using sage to compute $E(e_{K-1}^2)$ gives me

$$E(e_{K-1}^2) = K^2 v^2 + (K^2 - 2K + 1)e^2 + \cancel{2(K^2-K)ev} + (K^2 - 2K + 1)w^2 + 2(\cancel{(K^2-2K+1)e} + (K^2-K)v)w$$

$$E(e_K^2) = K^2 R + (K^2 - 2K + 1) E(e_{K-1}^2) + (K^2 - 2K + 1) Q + 2(K^2 - K) M$$

$$E(e_\infty^2) = -\frac{4K^2 - 4K + 1}{K^2 - 2K}$$

using K_∞ from a)

$$\boxed{E(e_\infty^2) \approx 0.0652}$$

c) Notice that $V_K = w_{K-1} \cancel{+ \delta_v}$ ~~where $\delta_v \sim (0, 0)$~~,

$$E[V_K W_{K-1}] = E[w_{K-1}^2] = 1 = M$$

suppose $w_K = \delta_w$ where $\delta_w \sim (0, Q)$

$$\begin{bmatrix} x_K \\ w_K \\ v_K \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{K-1} \\ w_{K-1} \\ v_{K-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_w \\ 0 \end{bmatrix}$$

$$y_K = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_K \\ w_K \\ v_K \end{bmatrix}$$

This form satisfies P7.2, hence we know that it will also have $P_\infty^+ = 0$

7.4 a) square root of $Q = I \Rightarrow \boxed{Q^{1/2} = I}$

b) $Q = \begin{bmatrix} H & HF \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ $F = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ $H = [1 \ 1]$

since $\text{rank}(Q) = 1$, the system is ~~Not~~ observable

c) The system is stable, $\therefore$ it is detectable.

d) Let $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$

$Q = \begin{bmatrix} F & FG \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & \frac{1}{2}G_{11} & \frac{1}{2}G_{12} \\ 0 & 1/2 & \frac{1}{2}G_{21} & \frac{1}{2}G_{22} \end{bmatrix} \rightarrow$ will always have rank 2

$\therefore$ (F, G) is controllable.

e) Since the system is controllable, it is also stabilizable

f) Both detectable and stabilizable $\therefore$ (unique) 1 positive definite sol'n.

g) the ss KF is stable

7.5 Prove $H = \begin{bmatrix} F^{-T} & F^{-T}H^TR^{-1}H \\ QF^{-T} & F+QF^{-T}H^TR^{-1}H \end{bmatrix}$ is symplectic.

a) $J^{-1}H^TJ = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} H^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} F^{-1} & F^{-1}Q \\ H^TR^{-1}HF^{-1} & F^T+H^TR^{-1}HF^{-1}Q \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ note: R and Q are symmetric

$= \begin{bmatrix} -H^TR^{-1}HF^{-1} & -F^T-H^TR^{-1}HF^{-1}Q \\ F^{-1} & F^{-1}Q \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$= \begin{bmatrix} F^T+H^TR^{-1}HF^{-1}Q & -H^TR^{-1}HF^{-1} \\ -F^{-1}Q & F^{-1} \end{bmatrix}$

Multiplying $J^{-1}H^TJ * H$

$= \begin{bmatrix} F^TF^{-T} + \cancel{H^TR^{-1}HF^{-1}QF^{-T}} - \cancel{H^TR^{-1}HF^{-1}QF^{-T}} & 0 \\ \cancel{-F^{-1}QF^{-T} + F^{-1}QF^{-T}} \to 0 & \cancel{-F^{-1}QF^{-T}H^TR^{-1}H} + F^{-1}F + \cancel{F^{-1}QF^{-T}H^TR^{-1}H} \end{bmatrix}$

$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$\therefore J^{-1}H^TJ = H^{-1}$

~~b) i) None of eigenvalues are 0~~
~~ii) The determinant is ±1.~~
~~Note: If det = ±1, then no eigenvalues are 0.~~
~~If 1 eigenvalue is 0, det becomes 0.~~
~~iii) If λ is an eigenvalue, so is $1/\lambda$~~
~~Note: The product of n eigenvalues of a matrix is the determinant of the matrix.~~
~~$\det = \lambda_1\lambda_2\lambda \dots \lambda_n = \lambda_1\lambda_2 \dots \frac{1}{\lambda_1}\frac{1}{\lambda_2} = 1$~~
~~$\therefore$ if I can prove iii), i) and ii) will follow~~
~~The eigenvalue~~

7.6 α-β filter. find α as function of β

$P = P^+$; $M = P^-$

$x_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} w_{k-1}$

$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$

~~(a) $M = FPF^T + Q$~~

~~$= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix} + Q$~~

~~$= \begin{bmatrix} P_{11}+TP_{12} & P_{12}+TP_{22} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix} + Q$~~

~~$= \begin{bmatrix} P_{11}+2TP_{12}+T^2P_{22} & \\ P_{21} & \end{bmatrix}$~~

a) $M = FPF^T + Q$ Note $F^{-1} = \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix}$

$P = F^{-1}(M-Q)F^{-T}$

$= \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_{11}-Q_{11} & M_{21}-Q_{21} \\ M_{12}-Q_{12} & M_{22}-Q_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -T & 1 \end{bmatrix}$

$= \begin{bmatrix} (M_{11}-Q_{11})-T(M_{12}-Q_{12}) & (M_{21}-Q_{21})-T(M_{22}-Q_{22}) \\ M_{12}-Q_{12} & M_{22}-Q_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -T & 1 \end{bmatrix}$

$= \begin{bmatrix} M_{11}-Q_{11}-2T(M_{12}-Q_{12})+T^2(M_{22}-Q_{22}) & (M_{21}-Q_{21})-T(M_{22}-Q_{22}) \\ M_{12}-Q_{12}-T(M_{22}-Q_{22}) & M_{22}-Q_{22} \end{bmatrix}$

Note $Q = \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix} \sigma^2$

$P_{11} = M_{11} - \frac{T^4}{4}\sigma^2 - 2TM_{12} + 2T\frac{T^3}{2}\sigma^2 + T^2M_{22} - T^2T^2\sigma^2$

$= M_{11} - 2TM_{12} + T^2M_{22} - T^4/4\,\sigma^2$ //

$P_{12} = M_{12} - \frac{T^3}{2}\sigma^2 - TM_{22} + TT^2\sigma^2$

$= M_{12} - TM_{22} + \frac{T^3}{2}\sigma^2$ //

$P_{22} = M_{22} - T^2\sigma^2$ //

b) $P = (I-KH)M = (I - MH^T(HMH^T+R)^{-1}H)M$

$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + V_k^2 \right)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}$

$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} (M_{11}+V_k^2)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}$

$= \begin{bmatrix} 1 - \frac{M_{11}}{M_{11}+V_k^2} & 0 \\ -\frac{M_{12}}{M_{11}+V_k^2} & 1 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}$

$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} M_{11} - \frac{M_{11}^2}{M_{11}+V_k^2} & M_{12} - \frac{M_{11}M_{12}}{M_{11}+V_k^2} \\ M_{12} - \frac{M_{11}M_{12}}{M_{11}+V_k^2} & M_{22} - \frac{M_{12}^2}{M_{11}+V_k^2} \end{bmatrix}$ //

More simply

$P = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}$

$= \begin{bmatrix} 1-K_1 & 0 \\ -K_2 & 1 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}$

$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} M_{11} - K_1M_{11} & M_{12} - K_1M_{12} \\ M_{12} - M_{11}K_2 & M_{22} - M_{12}K_2 \end{bmatrix}$ //

c) equating P_{22} $\quad M_{22} - T^2\sigma^2 = M_{22} - M_{12}K_2$

$M_{12}K_2 = T^2\sigma^2$

equating P_{12} $\quad M_{12} - TM_{22} + \frac{T^3}{2}\sigma^2 = M_{12} - K_1M_{12}$

$M_{12}K_1 = TM_{22} - \frac{T^3}{2}\sigma^2$

equating P_{11} $\quad M_{11} - 2TM_{12} + T^2M_{22} - \frac{T^4}{4}\sigma^2 = M_{11} - K_1M_{11}$

$M_{11}K_1 = 2TM_{12} - T^2M_{22} + \frac{T^4}{4}\sigma^2$

d) $K = \frac{1}{P_{11}^- + R}\begin{bmatrix} P_{11} & P_{12} \end{bmatrix}^T = \frac{1}{M_{11}+V_K^2}\begin{bmatrix} M_{11} & M_{12} \end{bmatrix}^T$

$K_1 = \frac{M_{11}}{M_{11}+V_K^2} \qquad K_2 = \frac{M_{12}}{M_{11}+V_K^2}$

$K_1 M_{11} + K_1 V_K^2 = M_{11} \qquad M_{12} = (M_{11}+V_K^2)K_2$

$K_1 V_K^2 = (1-K_1)M_{11} \qquad = \left(\frac{K_1 V_K^2}{1-K_1} + V_K^2\right)K_2$

$M_{11} = \frac{K_1 V_K^2}{1-K_1} \qquad M_{12} = \left(\frac{V_K^2}{1-K_1}\right)K_2$

e) Using $M_{12} = \frac{V_K^2 K_2}{1-K_1}$ and $M_{12}K_2 = T^2\sigma^2$

① $\frac{V_K^2 K_2^2}{1-K_1} = T^2\sigma^2 \Rightarrow V_K^2 K_2^2 = (1-K_1)T^2\sigma^2$ //

Using $M_{12}K_1 = TM_{22} - \frac{T^3}{2}\sigma^2$ and $M_{12} = \frac{V_K^2 K_2}{1-K_1}$

② $\frac{V_K^2 K_1 K_2}{1-K_1} = TM_{22} - \frac{T^3}{2}\sigma^2$

Using $M_{11}K_1 = 2TM_{12} - T^2 M_{22} + \frac{T^4}{4}\sigma^2$

③ $\frac{K_1^2 V_K^2}{1-K_1} = \frac{2T V_K^2 K_2}{1-K_1} - T^2 M_{22} + \frac{T^4}{4}\sigma^2$

using ② and ③

$\frac{K_1^2 V_K^2}{1-K_1} = \frac{2T V_K^2 K_2}{1-K_1} - \frac{T V_K^2 K_1 K_2}{1-K_1} - \frac{T^4}{4}\sigma^2$

④ $K_1^2 V_K^2 = 2T V_K^2 K_2 - T V_K^2 K_1 K_2 - \frac{T^4}{4}\sigma^2(1-K_1)$ //

f) $K_1 = \alpha \quad K_2 = \beta/T$

using ① and ④

$K_1^2 V_K^2 = 2T V_K^2 K_2 - T V_K^2 K_1 K_2 - \frac{T^2}{4} V_K^2 K_2^2$

Assume $V_K^2 = 1$

$\alpha^2 - 2\beta + \alpha\beta + \beta^2/4 = 0$

$\alpha = \pm\sqrt{2\beta} - \frac{\beta}{2}$ //

7.7 a) None of the eigenvalues are 0

$J^{-1}H^T J = H^{-1}$ means H^{-1} exists. Since H^{-1} exist, H is nonsingular $\therefore$ none of the eigenvalues are 0.

b) If λ is an eigenvalue, then so is $1/\lambda$

$|\lambda I - H^{-1}| = |\lambda I - J^{-1}H^T J| = |J^{-1}(\lambda I - H^T)J|$

$= |J^{-1}||\lambda I - H^T||J| = |\lambda I - H|$

This can only be satisfied if both λ and $1/\lambda$ are H's eigenvalues.

c) $|H| = \pm 1$

$|H^{-1}| = |J^{-1}H^T J| = \cancel{|J^{-1}|}|H^T|\cancel{|J|}$

$\frac{1}{|H|} = |H^T| = |H| \Rightarrow |H|^2 = 1 \Rightarrow |H| = \pm 1$

7.8 Since (F,H) is observable

$Q = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \end{bmatrix}$ $Qx \,!= 0$ for $x \,!= 0$

$\Rightarrow Hx = 0 \quad HF^2x = 0$

$HFx = 0$

Let $F' = (I-KH)F \quad H' = H$

$Q' = \begin{bmatrix} H \\ H(I-KH)F \\ H[(I-KH)F]^2 \\ \vdots \end{bmatrix}$ ① ② ③

from ① $Hx = 0$ only if $x == 0$

from ② $H(I-KH)Fx = 0$

since nonsingular $(I-HK)HFx = 0 \Rightarrow HFx = 0$

from ③ $H(I-KH)F(I-KH)Fx = 0$

$(I-HK)H(F^2x - FK(HFx)) = 0$

$\cancel{(I-HK)}HF^2x = 0 \Rightarrow HF^2x = 0$

and since $Hx, HFx, HF^2x, \ldots = 0$ (from (F,H) is observable), (F', H') is also observable!

7.9

Eq..

$\lambda = \frac{1}{R}$

$K_1 = -\frac{1}{8}\left(\lambda^2 + 8\lambda - (\lambda+4)*\sqrt{\lambda^2+8\lambda}\right)$

$P_{11}^+ = 1 = K_1 * R$

solving this 3 equations (using sage) gives me $R = \pm\frac{3}{16}\sqrt{17} + \frac{11}{16}$

$R = +\frac{3}{16}\sqrt{17} + \frac{11}{16} \approx 1.46$

(− side gives negative R)

$\boxed{R \le 1.46}$ //