Draft version. Downloaded from lukesy.net

13.1 $\dot{x} = -x + w = f(x,u,w)$ $w \sim (2, Q)$

$y = x + v = h(x,v)$ $v \sim (3, R)$

Let $x_0 = 0$, $w_0 = 2$, $v_0 = 3$

$\dot{x} = f(x_0,u_0,w_0) + \frac{\partial f}{\partial x}\Big|_0 (x-x_0) + \frac{\partial f}{\partial u}(u-u_0) + \frac{\partial f}{\partial w}(w-w_0)$

$y = h(x_0,v_0) + \frac{\partial h}{\partial x}\Big|_0 (x-x_0) + \frac{\partial h}{\partial v}(v-v_0)$

~~① $\begin{bmatrix}\dot{x}\\0\end{bmatrix} = \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\3\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}1\end{bmatrix} + w'$~~

~~$y = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x\\3\end{bmatrix} + v'$~~

$\dot{x} = 2 - x + 0 + w'$ where $w' \sim (0, Q)$

$y = 3 + x + v'$ where $v' \sim (0, R)$

~~either ① or~~ ② $x' = x + 3$ $\boxed{\begin{aligned}\dot{x}' &= -x' + 5 + w' \\ y &= x' + v'\end{aligned}}$

13.2 $\dot{x} = -x + u + w$ $w \sim (0, Q)$

$u \sim (u_0, 4)$

$\dot{x} = -x + u_0 + \Delta u + w$ where $\Delta u \sim (0,4)$

$E[(x+y-\bar{x}-\bar{y})(x+y-\bar{x}-\bar{y})]$ where $\bar{x}$ & $\bar{y}$ is 0

$= E[x^2] + 2\underbrace{E[xy]}_{\text{since uncorrelated}} {}^{\to 0} + E[y^2]$

$\Rightarrow \dot{x} = -x + u_0 + w'$ where $w' \sim (0, Q+4)$ and u_0 is perfectly known

13.3

a) x = constant scalar $y_k = \sqrt{x}(1+v_k)$ where $v_k \sim (0,R)$

$\hat{x}_k = y_k^2$

$E[\hat{x}_k - x] = E[(\sqrt{x}(1+v_k))^2 - x] = E[x(1+2v_k+v_k^2) - x]$

should be opposite

$= 2x E[v_k]^{\to 0} + xE[v_k^2]^{\to R}$

$= xR \Rightarrow -xR$

$E[(\hat{x}_k - x - xR)^2] = E[\{x + 2xv_k + xv_k^2 - x - xR\}^2]$

no need

$= E[4x^2v_k^2 + 2x^2v_k^3 - 2x^2v_kR + 2x^2v_k^3 + x^2v_k^4$

$-4x^2v_k^2R - \cancel{2x^2v_kR} - \cancel{x^2v_k^2R} + \cancel{x^2R^2}]$

$= 4x^2R + 0 + 0 + 0 + 3x^2R^2 - \cancel{x^2R^2} - \cancel{x^2R^2} + x^2R^2$

$= 4x^2R + 2x^2R^2 \Rightarrow 4x^2R + 3x^2R^2$

b) $E(x - \hat{x}_k) = E\left(x - \frac{1}{k}\sum_{i=1}^{k}\left(x(1+v_i)^2\right)\right)$

$= E\left(x - \frac{1}{k}\sum(x + 2xv_i + xv_i^2)\right)$

$= E\left(x - \frac{kx}{k} - \frac{2x}{k}\sum v_i - \frac{x}{k}\sum v_i^2\right)$ with $\sum v_i \to 0$, $\sum v_i^2 \to R$

$\boxed{= -\frac{xR}{k}}$

$E((x-\hat{x}_k)^2) = E\left(\left(-\frac{x}{k}\left(2\sum v_i - \sum v_i^2\right)\right)^2\right)$

$= E\left(\frac{x^2}{k^2}\left(4\sum_i\sum_j v_iv_j - 4\sum_i\sum_j v_iv_j^2 + \sum_i\sum_j v_i^2v_j^2\right)\right)$

not sure how to solve this but sol'n manuel ended up w/ this

$\frac{x^2}{k^2}\left(4Rk + 3R^2k + k(k-1)R^2\right)$

$= \boxed{\frac{4x^2R + (k+2)x^2R^2}{k}}$

As $k \to \infty$, variance $\to \boxed{x^2R^2}$

c) $f(x) = x \Rightarrow x_k = x_{k-1}$

$h(x) = \sqrt{x}(1+v) \Rightarrow y_k = \sqrt{x_k}(1+v_k)$

$F_{k-1} = 1$ $L_{k-1} = 0$

$P_k^- = P_{k-1}^+$

$x_k^- = x_{k-1}^+$

$H_k = (1+v_k)\frac{1}{2}(x_k)^{-1/2}$

$M_k = \sqrt{x_k}$

$K_k = P_k^- \frac{1+v_k}{2\sqrt{x_k}}\left(\frac{(1+v_k)^2}{4x_k}P_k^- + x_kR\right)^{-1}$

① $x_k^+ = x_k^- + K_k\left[y_k - \sqrt{x_k^-}\right]$

$= x_k^- + K_k\left[\sqrt{x}(1+v_k) - \sqrt{x_k^-}\right]$

② $P_k^+ = \left(1 - K_k\frac{1+v_k}{2\sqrt{x_k}}\right)P_k^-$

from ① $x_\infty = x_\infty + K_k\left[\sqrt{x}(1+v_k) - \sqrt{x_\infty}\right]$

$\Rightarrow x_\infty = x(1+v_k)^2$

from ② $P_\infty = \left(1 - K_\infty\frac{1+v_k}{2\sqrt{x_k}}\right)P_\infty \Rightarrow \left(1 - \frac{P_\infty\frac{(1+v_k)^2}{4x_k}}{P_\infty\frac{(1+v_k)^2}{4x_k} + x_kR}\right)P_\infty$

$\frac{x_kR}{\frac{(1+v_k)^2}{4x_k}P_\infty + x_kR} = 1 \Rightarrow P_\infty = 0$

Draft version. Downloaded from lukesy.net

13.4 $x_{k+1} = x_k + w_k$

$$y_k = x_k + v_k^2$$

$F_{k-1} = 1 \quad L_{k-1} = 1$

$P_k^- = P_{k-1}^+ + Q_{k-1}$

$\hat{x}_k^- = \hat{x}_k^+$

$$H_k = \frac{\partial h}{\partial x}\bigg|_{\hat{x}_k^-} = 1$$

$$M_k = \frac{\partial h}{\partial v}\bigg|_{\hat{x}_k^-} = 2v\Big|_{\hat{x}_k, 0} = 0$$

$$K_k = \frac{P_k^- 1}{P_k^- + 0} = 1$$

$P_k^+ = 0$

$\hat{x}_k^+ = \hat{x}_k^- + 1*(y_k - \hat{x}_k^-) = y_k$

a) $E(x_k - \hat{x}_k^+) = E(x_k - x_k - v_k^2) = \boxed{-R}$

b) modify y_k such that y_k' mean is 0

$y_k' = x_k + v_k^2 - R \quad ; \quad v_k' = v_k^2 - R$

$E(v_k'^2) = E((v_k^2 - R)^2) = E(v_k^4 - 2Rv_k^2 + R^2)$

Given v_k is uniform and variance is R

$$E(v_k^2) = R = \frac{1}{2c}\int_{-c}^{c} v_k^2 \, dv_k = \frac{v_k^3}{3}\bigg|_{-c}^{c} \frac{1}{2c}$$

$$= \frac{2c^3}{3*2c} \Rightarrow c = \sqrt{3R}$$

$$E(v_k^4) = \frac{1}{2c}\int_{-c}^{c} v_k^4 \, dv_k = \frac{v_k^5}{5}\bigg|_{-c}^{c}\frac{1}{2c} = \frac{\cancel{2}c^{\cancel{5}4}}{5*\cancel{2c}}$$

$$E(v_k'^2) = E(v_k^4) - 2R\,E(v_k^2) + R^2$$

$$= \frac{(\sqrt{3R})^4}{5} - 2R^2 + R^2 = \frac{9R^2}{5} - R^2 = \boxed{\frac{4R^2}{5}}$$

13.5 I do not understand the question, below is the answer according to the soln. manual.

$x_k = (-1)^k$

$y_k = 4(-1)^k$

13.6 $x_{k+1} = x_k^2 + w_k$ — w_k = zero mean. x_0 = uniformly dist. 0~1

$$E(x_0) = \int_0^1 x\,dx = \frac{x^2}{2}\bigg|_0^1 = 0.5$$

$$E(x_1) = E(x_0^2 + w_k) = E(x_0^2) = \int_0^1 x^2 dx = \frac{x^3}{3}\bigg|_0^1$$

$$= \boxed{1/3}$$

$$F_{k-1} = \frac{\delta f_{k-1}}{\delta x}\bigg|_{\hat{x}_{k-1}^+} = 2x_{k-1}\bigg|_{\hat{x}_{k-1}^+} = 2\hat{x}_{k-1}^+$$

$$L_{k-1} = \frac{\delta f_{k-1}}{\delta w}\bigg|_{\hat{x}_{k-1}^+} = 1$$

$\hat{x}_k^- = f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0)$

$\hat{x}_1^- = f_0(\hat{x}_0^+, 0, 0) = \boxed{1/4}$

13.7 Terminal velocity occurs when $\dot{x}_2 = 0$

$$\dot{x}_2 = \rho_0 e^{-\frac{x_1}{k}} \frac{x_2^2}{2x_3} - g + w_2 = 0$$

$$x_2 = \sqrt{\frac{g\,2x_3\,e^{\frac{x_1}{k}}}{\rho_0}} = \boxed{6939.6 \text{ ft/sec}}$$

13.8 $\dot{x} = f(x) + w \quad w \sim N(0, Q)$

$y_k = h(x_k) + v_k \quad v_k \sim N(0, R)$

$\hat{x}_k = a + b y_k + c y_k^2$

a) unbiased estimate means $E(x - \hat{x}_k) = 0$

note: $x \sim N(0, P_x)$

$$E(x - \hat{x}_k) = E(x) - E(a + b y_k + c y_k^2)$$

$$= 0 - a - b\,E(h(x_k) + v_k) - c\,E((h(x_k) + v_k)^2)$$

$$= -a - b\,E(h(x_k)) - b\,\cancel{E(v_k)}^{0} - c\,E(h(x_k)^2)$$

$$- 2c\,\underbrace{\cancel{E(h(x_k)v_k)}^{0}}_{\text{uncorrelated}} - c\,E(v_k^2)$$

$$= -a - b\,E(h(x_k)) - c\,E(h(x_k)^2) - cR = 0$$

$$\boxed{\Rightarrow a + b\,E(h(x_k)) + c\,E(h(x_k)^2) + cR = 0}$$

Draft version. Downloaded from lukesy.net

13.8 b) Find a, b, c so that $\hat{x}_k$ is minimum var est.
assume $h(x)$ is odd

$$P = E\left((x-\hat{x}_k)^2\right) = E\left(\left(x-a-b(h+v)-c(h^2+v^2)\right)^2\right)$$
$$= E(x^2) - 2aE(x) - 2b\,E(x(h+v)) - 2c\,E(x(h^2+v^2))$$
$$+a^2 + 2ab\,E(h+v) + 2ac\,E(h^2+v^2) + b^2 E((h+v)^2)$$
$$+2bc\,E((h+v)(h^2+v^2)) + c^2 E((h^2+v^2)^2)$$

$$\frac{\partial P}{\partial a} = -\cancel{2E(x)}^{\,0} + 2a + 2b\,\cancel{E(h+v)}^{\,0} + 2c\,E(h^2+v^2) = 0$$
h is odd, V is zero mean

$$= 2a + 2c\,E(h^2+v^2) = 0$$

$$\frac{\partial P}{\partial b} = -2E(x(h+\overset{\text{uncorr}}{\cancel{v}})) + 2a\,\cancel{E(h+v)}^{\,0} + 2b\,E(h^2 + \overset{\text{uncorr}\to 0}{\cancel{2hv}} + v^2)$$
$$+ 2c\,E(\cancel{\underbrace{h^3}_{\text{odd func.}} + \underbrace{h^2v + hv^2}_{\text{odd func.}} + \underbrace{v^3}_{\text{odd func.}}})^{\,0}$$
$$= -2E(xh) + 2b\,E(h^2+v^2) = 0$$

$$\frac{\partial P}{\partial c} = -2E(\overset{\text{odd}}{\cancel{x h^2}} + \overset{\text{uncorr}}{\cancel{v^2}}) + 2a\,E(h^2+v^2) + 2b\,\cancel{E(\underbrace{h^3}_{\text{odd}} + \underbrace{h^2v + hv^2}_{\text{uncorr}} + \underbrace{v^3}_{\text{odd}})}^{\,0}$$
$$+ 2c\,E(h^4 + \underbrace{2h^2v^2}_{\text{uncorr}} + v^4)$$
$$= 2a\,E(h^2+v^2) + 2c\,E(h^4 + 2h^2v^2 + v^4) = 0$$

One possible sol'n is:
$$a = 0 \quad c = 0 \quad b = \frac{E(xh)}{E(h^2)+R}$$

13.9 For $P_k^- = 1$, $R = 1$, $H = 3$
$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$
$$P_k^+ = (I - K_k H_k) P_k^-$$
$$K_k = 1 * 3(9*1+1)^{-1} = 3/10$$
$$P_k^+ = (1 - 3/10(3))\,1 = \underline{1/10}$$

For $H=2$: $K_k = 1*2(4*1+1)^{-1} = 2/5$
$$P_k^+ = (1 - 2/5(2))\,1 = \underline{1/5}$$

For $H=1$ $K_k = 1*1(1*1+1)^{-1} = 1/2$
$$P_k^+ = (1 - 1/2(1))\,1 = \underline{1/2}$$

13.10 $y_k = x_k^2 + v_k \Rightarrow \frac{\delta h_k}{\delta x} = 2x_k$, $\frac{\delta h_k}{\delta v} = 1$
$\hat{x}_k^- = 1$, $x_k = 5$, $y_k = 25$, $P_k^- = 1$, $R_k = 4$

$\hat{x}_{k,0}^+ = \hat{x}_k^- = 1$ $P_{k,0}^+ = P_k^- = 1$

1st: $H_x = \frac{\delta h_k}{\delta x}\Big|_{\hat{x}_{k,0}^+} = 2\hat{x}_{k,0}^+ = 2$; $M_k = 1$; $K_k = \frac{1*2}{1*2^2+4} = \frac{1}{4}$; $\hat{x}_{k,1}^+ = 1 + 1/4(25-1)$
$\hat{x}_{k,1}^+ = \underline{7}$

2nd: $P_{k,1}^+ = (1 - \frac{1}{4}(2))\,1 = 1/2$
$H_x = \frac{\delta h_k}{\delta x}\Big|_{\hat{x}_{k,1}^+} = 14$; $M_k = 1$; $K_k = \frac{1/2(14)}{1/2(14)^2+4} = \frac{7}{102}$; $\hat{x}_{k,2}^+ = 7 + \frac{7}{102}(25-49)$
$\hat{x}_{k,2}^+ = \underline{5.35}$

13.11 Prove lemma 6 for scalar RV x
Prove: $x \sim N(0, P)$
$$E[x\,Tr(Axx^T)] = 0 \quad\text{and}\quad E[Tr(Axx^TBxx^T)] = 2Tr(APBP) + Tr(AP)Tr(BP)$$
i) $E[x\,Tr(Axx^T)] = E[Ax^3] = \underline{0}$ note $E[x^3] = 0$

ii) $E[Tr(Axx^T)\,Tr(Bxx^T)] = E[ABx^4] = \underline{AB3P^2}$
where as $2Tr(APBP) + Tr(AP)Tr(BP) = \underline{3ABP^2}$ same

13.12 $\dot{x} = x^2 + w$, $\hat{x}_k^+ = 0$
i) 1st order EKF
$$\dot{\hat{x}} = f(\hat{x}_k, u, 0) = \hat{x}^2 = \underline{0 \text{ for } \hat{x} = \hat{x}_k^+}$$

ii) 2nd order EKF
$$\dot{\hat{x}} = f(\hat{x}, u, 0, t) + \frac{1}{2}\sum_{i=1}^{n} \phi_i Tr\left[\frac{\partial^2 f_i}{\partial x^2}\Big|_{\hat{x}} P\right]$$
$$= \hat{x}^2 + \frac{1}{2}(2P) = \underline{P} \text{ for } \hat{x} = \hat{x}_k^+$$

13.13 $y_k = x_k^2 + v_k$ $v_k \sim (0, R)$
$P_k^- = 1$, $\hat{x}_k^- = 1$ is unbiased

a) 1st order
$$H_k = \frac{\delta h_k}{\delta x} = 2x_k \Rightarrow \Big|_{\hat{x}_k^- = 1} = 2$$
$$M_k = \frac{\delta h_k}{\delta v} = 1$$
$$K_k = \frac{1*2}{1*2^2+R} = \frac{2}{4+R}$$
$$\hat{x}_k^+ = \hat{x}_k^- + K_k(y_k - h_k(\hat{x}_k^-, 0, t_k))$$
$$= 1 + \frac{2}{4+R}(x_k^2 + v_k - 1)$$
Given $\hat{x}_k^-$ is unbiased
$$E(x_k - \hat{x}_k^-) = 0 \Rightarrow E(x_k) = 1$$
$$E((x_k - \hat{x}_k^-)^2) = 1 \Rightarrow E(x_k^2) - 2E(x_k) + 1 = 1 \Rightarrow E(x_k^2) = 2$$
$$E(\hat{x}_k^+) = E\left(1 + \frac{2}{4+R}(x_k^2 + v_k - 1)\right) = \boxed{1 + \frac{2}{4+R}}$$

b) 2nd order
$$\Pi_k = \frac{1}{2}K_k \sum_{i=1}^{m} \phi_i Tr\left[\frac{\delta^2 h_i(x_k, t_k)}{\delta x^2}\Big|_{\hat{x}_k^-} P_k^-\right]$$
$$= \frac{1}{2}\left(\frac{2}{4+R}\right) 1 * [2*1] = \frac{2}{4+R}$$
$$\hat{x}_k^+ = \hat{x}_k^- + K_k[y_k - h(\hat{x}_k^-)] - \Pi_k$$
$$E(\hat{x}_k^+) = 1 + \frac{2}{4+R} - \frac{2}{4+R} = \boxed{1}$$

Draft version. Downloaded from lukesy.net

13.14 $z_{k+1} = a z_k + w_k$ $\quad w_k \sim (0, Q)$

$y_k = z_k + v_k$ $\quad v_k \sim (0, R)$

~~Assume $w_p = 0$~~

Let $x'_{k+1} = \begin{bmatrix} z_{k+1} \\ a_{k+1} \end{bmatrix}$, $\begin{bmatrix} z_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} a_k z_k + w_k \\ a_k + w_p \end{bmatrix}$

$$y'_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x'_k + v_k$$

Time Update

$$F_{k-1} = \frac{\delta f_{k-1}}{\delta x}\bigg|_{\hat{x}^+_{k-1}} = \begin{bmatrix} a_k & z_k \\ 0 & 1 \end{bmatrix}\bigg|_{\hat{x}^+_{k-1}} = \begin{bmatrix} a_k & \hat{x}^+_{k-1} \\ 0 & 1 \end{bmatrix}$$

$$L_{k-1} = \frac{\delta f_{k-1}}{\delta w}\bigg|_{\hat{x}^+_{k-1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since we are told to assume $w_p = 0$ and the variance for $a = 0$ at steadystate. Time update:

$$P^-_k = \begin{bmatrix} P^-_{k,11} & 0 \\ 0 & 0 \end{bmatrix} = F_{k-1} \begin{bmatrix} P^+_{k-1,11} & 0 \\ 0 & 0 \end{bmatrix} F^T_{k-1} + L_{k-1} Q_{k-1} L^T_{k-1}$$

$$= \begin{bmatrix} a_k^2 P^+_{k-1,11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \hat{z}^-_k \\ \hat{a}^-_k \end{bmatrix} = \begin{bmatrix} \hat{a}^+_{k-1} \hat{z}^+_{k-1} \\ \hat{a}^+_{k-1} \end{bmatrix}$$

Measurement Update:

$$H_k = \frac{\delta h_k}{\delta x}\bigg|_{\hat{x}^-_k} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad M_k = 1$$

$$K_k = \begin{bmatrix} P^-_{k,11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P^-_{k,11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + R_k \right)^{-1}$$

$$= \begin{bmatrix} P^-_{k,11} \\ 0 \end{bmatrix} \left(P^-_{k,11} + R_k \right)^{-1} = \begin{bmatrix} \frac{P^-_{k,11}}{P^-_{k,11} + R_k} \\ 0 \end{bmatrix}$$

$$P^+_k = \left(I - \begin{bmatrix} \frac{P^-_{k,11}}{P^-_{k,11} + R_k} \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} P^-_{k,11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{P^-_{k,11}}{P^-_{k,11} + R_k} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^-_{k,11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{R_k P^-_{k,11}}{P^-_{k,11} + R_k} & 0 \\ 0 & 0 \end{bmatrix}$$

Notice how the upper left element of K_k and P^+_k and P^-_k is similar to a linear KF.